Rapid Communications

Hidden thermal structure in Fock space

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The emergence of quantum statistical mechanics from individual pure states of closed many-body systems is under intensive investigation. While most effort has been put on the impact of the *direct* interaction (i.e., the usual mutual interaction), here we study systematically and analytically the impact of the *exchange* interaction that arises from the particle indistinguishability. We show that this interaction leads an overwhelming number of Fock states to exhibit a structure that can be resolved only by observables adjusted according to the system's dynamical properties and from which thermal distributions emerge. This hidden thermal structure in Fock space is found to be related to the so-called limit shape of random geometric objects in mathematics. The structure enables us to uncover, for both ideal and nonideal Fermi gases, a striking mechanism for the emergence of quantum statistical mechanics from individual eigenstates.

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There has been increasing evidence [1–8] showing that a closed quantum many-body system can act as its own heat bath, leading to the emergence of equilibrium statistical mechanics from pure states (see Refs. [9–12] for reviews). Notwithstanding this, the ingredients indispensable for this emergence remain an open problem. It has been shown that the direct interaction, via driving many-body quantum chaos, gives rise to complex structures of the eigenstates, from which the Fermi-Dirac (FD) or Bose-Einstein (BE) distribution arises [3,10,13,14]. The need of this interaction conforms to standard statistical mechanics [15], whereas the studies of entanglement entropy suggest that without the direct interaction, a thermal distribution arises also [16–19]. That thermal distributions exist in such a broad range of extreme conditions motivates one to explore universal routes to their emergence from pure states. Furthermore, the exchange interaction—"a peculiar mutual effect of particles that are in the same quantum state" [15]—is a common ingredient of quantum many-body systems. It is a building block of traditional ensemble-based quantum statistical mechanics, giving rise to the FD (BE) distribution and many intriguing phenomena ranging from the BE condensation to the Haldane-Wu fractional exclusion statistics [20,21]. Thus in-depth investigations of the exchange interaction in the emergence of statistical mechanics from pure states are of fundamental importance and are urgently needed.

In this Rapid Communication, we study systematically and analytically how the exchange interaction drives thermal equilibrium phenomena at individual pure states. For simplicity, we focus on Fermi statistics, and consider N ($\gg$1) indistinguishable fermions confined in a volume [22] for both situations: with and without the direct interaction. Without the direct interaction, an ideal Fermi gas results; the exchange interaction endows it with a many-body nature. Its eigenstate is a Fock state λ, represented by a pattern of the number of particles occupying a single-particle eigenstate. Three classes of representative single-particle eigenstates are considered, corresponding to distinct quantum motions (Fig. 1): Liouville integrable, chaotic, and Anderson localized. To realize the first, we put the particle on a torus (a1) or in a one-dimensional harmonic potential (a2), the second in a chaotic cavity (b), and the third in a quasi-one-dimensional cavity with scatterers randomly place inside (c). When the direct interaction is switched on, a nonideal Fermi gas results, whose eigenstate Φ is a superposition of Fock states. In this Rapid Communication, we first uncover a thermal structure hidden in Fock space, and then study its consequences on both ideal and nonideal Fermi gases.

The main results are summarized as follows:

First, we find that, irrespective of dynamical properties (Liouville integrable, chaotic, or Anderson localized) of single-particle motion, for an overwhelming number of Fock states the FD distribution emerges from an individual occupation number pattern (cf. Table I), and can be resolved only by appropriate observables (that is, this emergence does not ensure that in a given Fock state, the expectation values of all observables are thermal). As such, this is a hidden thermal structure. Moreover, it has nothing to do with many-body quantum chaos, but is related to the limit shape of random geometric objects [23–27], a subject well explored by mathematicians.

Second, we find that the influence of dynamical properties is to determine whether an observable can resolve the thermal structure. Table I gives the results for the one-particle

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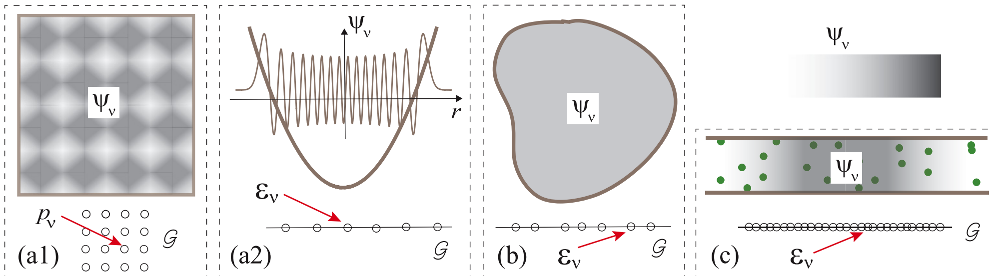

FIG. 1. Schematic representation of the spatial structures of single-particle eigenstate ψ_ν and the good quantum number space $\mathcal{G}$ (open circles) of Liouville integrable [(a1), (a2)], chaotic (b), and Anderson localized (c) motions in distinct setups. Green dots in (c) are scatterers.

correlation function $M_{rr'}$ between two spatial points r, r'. It shows that the short-ranged (small $|r-r'|$) correlation is always thermal, implying that if a subsystem is small, an individual λ, namely, a many-body eigenstate of ideal Fermi gas, acts as the heat bath of the subsystem, irrespective of dynamical properties. This is in spirit consistent with the results for the reduced density matrix based on the *canonical typicality* [5–7], which makes no reference to the system's constructions, whereas the long-ranged (large $|r-r'|$) correlation is thermal only if the single-particle motion is chaotic. In this case λ acts as the heat bath of the entire system.

Third, we find that, without Berry's conjecture [3,29], the eigenstate Φ of nonideal Fermi gases on a torus exhibits *eigenstate thermalization* [3]. Specifically, we show [Eq. (16)] that the short-ranged correlation at Φ is thermal, i.e., governed by the FD distribution, but not the detailed constructions of Φ.

Our findings suggest that the thermal structure hidden in the Fock space, arising from the exchange interaction, namely, the particle indistinguishability, is a basis of the emergence of thermal equilibrium phenomena from pure states. In particular, they indicate a striking mechanism for the eigenstate thermalization, different from those reported in literatures [1–3,8,9].

Observable-resolved structure $\Lambda(\lambda)$ *of individual* λ. An individual Fock state λ is a pattern $\{n_\nu\}$, where n_ν $(=0,1)$ is the occupation number at the single-particle eigenstate ψ_ν. ν denotes the complete set of good quantum numbers associated with the single-particle motion, which refers to the eigenmomentum p_ν for free motion (a1) and to the eigenenergy ε_ν for harmonic oscillation (a2), chaotic motion (b), and Anderson localization (c). Given a system, all ν constitute a space, denoted as $\mathcal{G}$ (Fig. 1). We will show that there are intimate relations between resolving the fine structures of $\{n_\nu\}$ by observables and the emerging of FD distribution from individual many-body eigenstates of ideal or nonideal Fermi gases. To this end we first illustrate in this part how distinct observables resolve structures of individual λ at different scales of $\mathcal{G}$.

We take a family of basic observables, namely, the one-particle correlation function $M_{rr'}$ at different ranges of $|r-r'|$. At λ the correlation function is

$$M_{rr'} \equiv \langle\lambda|c_{r'}^\dagger c_r|\lambda\rangle = \sum_\nu n_\nu C_\nu(r,r'). \tag{1}$$

Here, c_r ($c_r^\dagger$) is the annihilation (creation) operator at r, and $C_\nu(r,r') \equiv \psi_\nu(r)\psi_\nu^*(r')$ is the autocorrelation of $\psi_\nu(r)$.

(i) If $C_\nu(r,r')$ varies slowly with ν (Fig. 2, left),

$$C_\nu(r,r') \approx C_{\nu'}(r,r'), \qquad \text{for nearest } \nu,\nu', \tag{2}$$

then $\mathcal{G}$ has a "natural" decomposition into many subspaces $\mathcal{G}_m$ (Fig. 2, left). [We are not aware of generic conditions for Eq. (2). Thus we will justify it and derive the conditions for distinct dynamical systems later.] In each $\mathcal{G}_m$, C_ν (r, r' fixed) and ε_ν are approximately a constant, denoted as $C_m(r,r')$ and ε_m, respectively, i.e.,

$$\mathcal{G} = \oplus_m \mathcal{G}_m, \quad \forall \nu \in \mathcal{G}_m : C_\nu(r,r') \approx C_m(r,r'), \quad \varepsilon_\nu \approx \varepsilon_m. \tag{3}$$

By Eq. (2) the number of elements of $\mathcal{G}_m$, denoted as G_m, is $\gg 1$ [30]. Using the decomposition (3) we obtain

$$\sum_\nu n_\nu C_\nu(r,r') = \sum_m C_m(r,r') \sum_{\nu\in\mathcal{G}_m} n_\nu. \tag{4}$$

TABLE I. Structures of the individual Fock state λ resolved by the spatial correlation function $M_{rr'}$ of distinct ranges.

Eigenstate	Short ranged [28]		Long ranged	
ψ_ν	Structure resolved	Expectation value	Structure resolved	Expectation value
Integrable (a1), (a2)	FD	Thermal [Eq. (11)]	$\{n_\nu\}$	Athermal [Eq. (1)]
Chaotic (b)	FD	Thermal [Eq. (11)]	FD	Thermal [Eq. (11)]
Localized (c)	FD	Thermal [Eq. (13)]	$\{n_\nu\}$	Athermal [Eq. (1)]

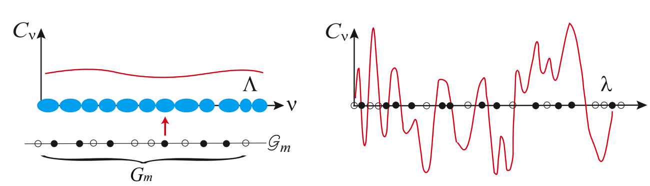

FIG. 2. Left: When C_ν varies slowly with ν, a decomposition of the space $\mathcal{G}$ into subspaces $\mathcal{G}_m$ (blue cells) results. Consequently, $M_{rr'}$ can resolve only the structure Λ less fine than λ. Right: When C_ν varies rapidly with ν, the decomposition does not follow and $M_{rr'}$ can resolve the fine structure of λ. Solid (open) circles denote (un)occupied eigenstates ν.

With the help of this result we reduce Eq. (1) to

$$M_{rr'} = \sum_m N_m C_m(r, r'), \quad N_m = \sum_{\nu\in\mathcal{G}_m} n_\nu. \tag{5}$$

Therefore, provided that Eq. (2) holds, $M_{rr'}$ cannot resolve n_ν at a specific ν; rather, it resolves a less fine structure $\{N_m\} \equiv \Lambda(\lambda)$, which is constrained by [31]

$$\sum_m N_m = N \quad \left(= \sum_\nu n_\nu\right),$$
$$\sum_m N_m \varepsilon_m \approx E \quad \left(= \sum_\nu n_\nu \varepsilon_\nu\right). \tag{6}$$

(ii) If $C_\nu(r, r')$ varies rapidly with ν (Fig. 2, right), then neither the decomposition (3) nor the reduction (5) follows. As $M_{rr'}$ is given by Eq. (1), a fine tuning in the pattern $\{n_\nu\}$ can lead to a significant change in $M_{rr'}$. That is, $M_{rr'}$ can resolve the fine structure of $\{n_\nu\}$.

Here, we make two remarks. First, the decomposition (3) resembles some ideas of von Neumann [32] regarding the fundamentals of the statistical mechanics of closed quantum systems, specifically, that observables can induce the decomposition of the space of quantum states. However, his decomposition refers to the Hilbert space spanned by the eigenstates of the entire system, which are λ for an ideal Fermi gas and Φ for a nonideal Fermi gas, whereas the decomposition (3) refers to $\mathcal{G}$. Second, although Λ looks similar to the "macroscopic state" of Landau [33], there are conceptual differences. Notably, as discussed, Λ is resolved only by proper observables, whereas the macroscopic state is independent of observables.

Emergence of thermal structures from $\Lambda(\lambda)$. A question naturally is as follows: What does the structure $\Lambda(\lambda)$ look like? To study this problem we note that by definition of Λ, distinct λ [constrained by Eq. (6)] can correspond to the same structure Λ. The number of λ corresponding to Λ is given by $\prod_m \frac{G_m!}{N_m!(G_m-N_m)!} \equiv W[\Lambda]$. From this expression we see that W has a sharp peak at some $\Lambda^* \equiv \{N_m^*\}$. Physically, this means that an overwhelming number of λ have the same observable-resolved structure Λ^*.

Now we can show that the thermodynamic relation emerges from an individual λ satisfying $\Lambda[\lambda] = \Lambda^*$: This is in contrast to standard statistical mechanics where thermodynamics is built upon an ensemble. By definition,

$$\left.\frac{\partial S}{\partial N_m}\right|_{\Lambda=\Lambda^*} = \alpha + \beta\varepsilon_m, \quad S \equiv \ln W[\Lambda]. \tag{7}$$

Here, α, β are the Lagrange multipliers. They depend on N, E, and so do N_m^* and $W[\Lambda^*]$. Taking this and Eqs. (6) and (7) into account, we find that

$$\left.\frac{\partial S}{\partial E}\right|_{\Lambda=\Lambda^*} = \frac{\partial}{\partial E}\sum_m N_m(\alpha + \beta\varepsilon_m)\Big|_{\Lambda=\Lambda^*} = \beta, \tag{8}$$

where in deriving the last equality we have used the fact that N, E are independent variables. Similarly, we have

$$\left.\frac{\partial S}{\partial N}\right|_{\Lambda=\Lambda^*} = \frac{\partial}{\partial N}\sum_m N_m(\alpha+\beta\varepsilon_m)\Big|_{\Lambda=\Lambda^*} = \alpha. \tag{9}$$

Thus $S = \ln W[\Lambda^*]$ gives the thermodynamic entropy [22], β the inverse thermodynamic temperature $\frac{1}{T}$, and $-\frac{\alpha}{\beta}$ the chemical potential μ. So Eqs. (8) and (9) reduce to $\frac{\partial S}{\partial E}|_{\Lambda=\Lambda^*} = \frac{1}{T}$ and $\frac{\partial S}{\partial N}|_{\Lambda=\Lambda^*} = -\frac{\mu}{T}$. Note that these relations are independent of the explicit form of Λ^*.

Furthermore, by substituting the explicit form of $W[\Lambda]$ into Eq. (7), we obtain

$$N_m^*/G_m = \left(e^{\frac{\varepsilon_m-\mu}{T}} + 1\right)^{-1} \equiv f_{\rm FD}(\varepsilon_m). \tag{10}$$

So Λ^* is determined by FD distribution $f_{\rm FD}$, i.e., is a thermal structure. Unlike standard textbooks [15], here $f_{\rm FD}$ refers to individual λ, not an ensemble. The emergence of FD (BE) distribution from pure states has recently appeared as a new fundamental aspect of statistical mechanics [14,16,17]. Most importantly, $f_{\rm FD}$ can be resolved only if Eq. (2) holds, whereas in textbooks thermal distributions have nothing to do with observables.

Probing the hidden thermal structure Λ^*. Let the Fock space constrained by Eq. (6) be equipped with a uniform probability measure, and λ be drawn randomly from this measure. Equation (5) and the analysis above suggest that $M_{rr'}$ has a *typical* value (with respect to this measure), because an overwhelming number of λ satisfy $\Lambda(\lambda) = \Lambda^*$. Combining Eqs. (5) and (10) we find that, provided that Eq. (2) holds, this typical value is

$$M_{rr'} = \int d\mu(\nu)\left(e^{\frac{\varepsilon_\nu-\mu}{T}} + 1\right)^{-1} C_\nu(r, r'). \tag{11}$$

Here, $d\mu(\nu)$ gives the number of single-particle eigenstates in $d\nu$. The left-hand side of Eq. (11) is the expectation value of $c_{r'}^\dagger c_r$ at λ, while the right-hand side is the thermal average of C_ν. Note that the latter is determined by thermodynamic quantities T, μ, and thus fine tunings of $\{n_\nu\}$ do not change the value of $M_{rr'}$. This implies that for an overwhelming number of (but not all) λ constrained by Eq. (6), $M_{rr'}$ takes the same value—the onset of eigenstate thermalization [2,3,8] of ideal Fermi gases. Moreover, Eq. (11) provides a guide for probing Λ^*.

For highly excited λ, the thermal de Broglie wavelength is much smaller than the mean distance between two nearest particles. So the FD distribution in Eq. (11) can be well approximated by the Maxwell-Boltzmann (MB) distribution. In the Supplemental Material [34] we show that the MB distribution appearing from $M_{rr'}$ results from the quantum entanglement of indistinguishable particles. Provided particles are *distinguishable*, this entanglement does not exist (since neither the exchange nor direct interaction exists), and unlike Eq. (11), the MB distribution cannot emerge from the one-particle correlation function [34]. This scenario is fundamentally from standard statistical physics: The former refers to a pure quantum state while the latter to an ensemble.

The remainder is to find the conditions under which Eq. (2) holds. Below, we consider the single-particle quantum motions in Fig. 1 separately, and show that *precisely at this point, dynamical properties make significant differences* (see Table I for a summary of the results below). In essence, distinct motions give rise to distinct spatial structures of ψ_ν (Fig. 1) and thus the autocorrelation $C_\nu(r, r')$ of ψ_ν displays distinct dependences on ν.

(a1) With the substitution of $\psi_\nu(r) = \frac{e^{ip_\nu\cdot r}}{L}$ (L the torus size), $C_\nu \sim e^{ip_\nu\cdot(r-r')}$. (i) For $|r-r'| \ll L$, since the difference between nearest neighbors $p_\nu, p_{\nu'}$ is $O(L^{-1})$, we have $|(p_\nu - p_{\nu'})\cdot(r-r')| \ll 1$. From this we find that $p_\nu \cdot (r-r')$ varies slowly with ν, and justify Eq. (2). Thus we have Eq. (11), i.e., the short-ranged $M_{rr'}$ is thermal. (ii) For $|r-r'| = O(L)$, we have $|(p_\nu - p_{\nu'})\cdot(r-r')| = O(1)$. Thus $p_\nu\cdot(r-r')$ varies rapidly with ν and Eq. (2) breaks down. So the long-ranged $M_{rr'}$ is given by Eq. (1), i.e., athermal, and cannot be used to probe Λ^*.

(a2) The eigenvalue $\varepsilon_\nu = \nu + \frac{1}{2}$ and corresponding eigenstate $\psi_\nu(r) = \frac{\pi^{-1/4}}{\sqrt{2^\nu \nu!}} e^{-\frac{r^2}{2}} H_\nu(r)$, where H_ν is the Hermite polynomial. For $N \gg 1$ most fermions occupy highly excited single-particle eigenstates. Thus the sum in Eq. (1) is dominated by large ν, for which $\psi_\nu(r) \sim \cos(\sqrt{2\varepsilon_\nu} r)$. Substituting this asymptotic expression into C_ν and repeating the discussions on (a1), we find that $M_{rr'}$ is thermal for $|r-r'| \ll \sqrt{E/N}$ and athermal otherwise.

(b) To calculate C_ν we consider (i) large and (ii) small ε_ν separately. For (i) we perform the Wigner transformation $C_\nu(r,r') \equiv \int dp e^{-i(r-r')\cdot p} \Psi_\nu(q,p)$ with $q \equiv \frac{1}{2}(r+r')$, and adopt Berry's conjecture for single-particle chaotic motion [29], $\Psi_\nu(q,p) = \frac{\delta[\varepsilon_\nu - H(q,p)]}{\iint dq dp \delta[\varepsilon_\nu - H(q,p)]}$, with H being the Hamiltonian. This conjecture implies that $|\psi_\nu(r)|$ is homogeneous on large scales. Unlike Ref. [3], here the conjecture is not made for many-particle motion. Using the conjecture we obtain $C_\nu \sim f(\frac{|r-r'|}{\lambda_{\varepsilon_\nu}})$, with $\lambda_{\varepsilon_\nu}$ being the de Broglie wavelength at energy ε_ν. The function $f(x)$ oscillates in x, whose explicit form is unimportant. For nearest ν, ν' and for any r, r', we have

$$|r-r'|\left(\lambda_{\varepsilon_{\nu'}}^{-1} - \lambda_{\varepsilon_\nu}^{-1}\right) \sim (|r-r'|/L)(\Delta/\varepsilon_\nu)^{1/2} \ll 1, \quad (12)$$

with Δ being the level spacing and L the cavity size. From this we find that $f(\frac{|r-r'|}{\lambda_{\varepsilon_\nu}})$ is the same for nearest ν, ν'. Thus Eq. (2) is justified. For (ii) we do not expect Berry's conjecture to hold, since it is based on the semiclassical approximation. So Eq. (2) breaks down in general. But, the number of particles occupying low-lying single-particle states is $\ll N$. Thus their contributions to the sum in Eq. (1) are negligible, and the breakdown of Eq. (2) has no effect on $M_{rr'}$. So both short- and long-ranged $M_{rr'}$ are thermal and Eq. (11) follows.

(c) The Anderson localization [35–37] implies that the eigenvalues $\{\varepsilon_\nu\}$ are discrete and dense, and ψ_ν exhibits exponential localization in the longitudinal direction (Fig. 1). Moreover, the localization center has a singular dependence on ν: As ν approaches ν', the distance between localization centers of ψ_ν and $\psi_{\nu'}$ diverges. In addition, the localization length varies with ν. As a result, (i) if $|r-r'|$ is sufficiently large, C_ν varies rapidly with ν. Thus Eq. (2) breaks down and the long-ranged $M_{rr'}$ is athermal. (ii) For r, r' in the same localization volume, the sum in Eq. (1) is dominated by the subset of $\{\varepsilon_\nu\}$ that corresponds to this volume. Since each localization volume is an effective chaotic cavity, we can repeat the analysis of (b). As a result, we obtain Eq. (11), but with $d\mu$ replaced by $d\mu_{\text{loc}}$ which gives the number of eigenstates in a localization volume and the interval $d\nu$,

$$M_{rr'} = \int d\mu_{\text{loc}}(\nu)\left(e^{\frac{\varepsilon_\nu - \mu}{T}} + 1\right)^{-1} C_\nu(r,r'). \quad (13)$$

So we can use the short-ranged correlation to probe Λ^*.

Different mechanism for eigenstate thermalization in nonideal Fermi gas. Now we switch on the direct hard-sphere interaction between particles. For simplicity, we consider particles on a torus. This system is essentially the same as what was studied in Ref. [3]. An eigenstate Φ of this system, corresponding to the eigenenergy E, is a superposition of $\lambda \in \Omega_{N,E}$, where $\Omega_{N,E}$ is composed of all λ satisfying $\sum_\nu n_\nu = N$ and $\sum_\nu n_\nu \varepsilon_\nu = E$ ($\varepsilon_\nu = \frac{p_\nu^2}{2}$),

$$|\Phi\rangle = \sum_{\lambda\in\Omega_{N,E}} C_\lambda |\lambda\rangle, \qquad \sum_{\lambda\in\Omega_{N,E}} |C_\lambda|^2 = 1. \quad (14)$$

Note that for this system the many-body eigenenergy E is exactly the total kinetic energy, and the direct interaction enters only into the coefficients C_λ. By simple algebra we find the one-particle correlation function at Φ,

$$\langle\Phi|c_{r'}^\dagger c_r|\Phi\rangle = \sum_{\lambda\in\Omega_{N,E}} |C_\lambda|^2 M_{rr'} + \sum_{\lambda,\lambda'\in\Omega_{N,E}} C_\lambda^* C_{\lambda'} \sum_{\nu\neq\nu'} \frac{e^{i(p_\nu\cdot r - p_{\nu'}\cdot r')}}{L^2} \langle\lambda|c_\nu^\dagger c_{\nu'}|\lambda'\rangle, \quad (15)$$

where $M_{rr'} = \frac{1}{L^2}\sum_\nu e^{ip_\nu\cdot(r-r')}\langle\lambda|c_\nu^\dagger c_\nu|\lambda\rangle$. The left-hand side on Eq. (15) is translationally invariant, i.e., depends on r, r' via $r-r'$, for the system has the translation symmetry. But this invariance is violated by the second term on the right-hand side. Thus this term must vanish, giving $\langle\Phi|c_{r'}^\dagger c_r|\Phi\rangle = \sum_{\lambda\in\Omega_{N,E}} |C_\lambda|^2 M_{rr'}$. On the other hand, for generic Φ, most weights of $|C_\lambda|^2$ go to λ satisfying $\Lambda(\lambda) = \Lambda^*$, because, as shown above, $W[\Lambda]$ has a sharp peak at Λ^* [38]. For these λ and corresponding $M_{rr'}$ we use the results for (a1) summarized in Table I. In particular, $M_{rr'}$ takes the typical value (11) for $|r-r'| \ll L$, with T, μ in Eq. (11) determined by N, E. As $M_{rr'}$ are insensitive to the fine structure of λ, it can be further pulled out of the sum $\sum_{\lambda\in\Omega_{N,E}}(\cdots)M_{rr'}$, giving

$$\langle\Phi|c_{r'}^\dagger c_r|\Phi\rangle = M_{rr'} \sum_{\lambda\in\Omega_{N,E}} |C_\lambda|^2 = M_{rr'} = \int \frac{dp}{(2\pi)^2} \frac{e^{ip\cdot(r-r')}}{e^{(p^2/2-\mu)/T} + 1}. \quad (16)$$

Thus the eigenstate thermalization is justified for short-ranged one-particle correlation at both low- and high-lying Φ. Note that, unlike Ref. [3], we did not use Berry's conjecture made for many-particle chaotic motion, which has not yet been proven.

Relations between Λ^* and the limit shape of random geometric objects. Finally, we wish to understand from more rigorous viewpoints why the thermal structure Λ^* can arise merely from the particle indistinguishability. Let us view $\lambda = \{n_\nu\}$ as a geometric object—a collection of "skyscrapers" located at ν wherever $n_\nu = 1$ (see, e.g., Fig. 3). It turns out that despite the shapes of such objects appear to be random,

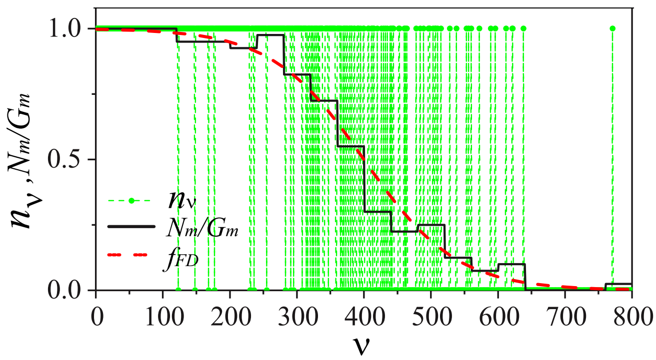

FIG. 3. A Fock state $\lambda = \{n_\nu\}$ of $N = 400$ fermions of total energy $E = 87\,800$ confined in a harmonic potential, or a partition of E into N distinct summands in number theory, is generated randomly in simulations. For the observable-resolved structure $\Lambda = \{N_m\}$ ($G_m = 40$) of a typical λ, N_m/G_m is fitted well by $f_{\rm FD}$ with $\mu = 400.1$ and $T = 68.84$.

as discovered by mathematicians [23–27], when they are large and rescaled properly, they can concentrate on a smooth and nonrandom limit shape.

To keep quantitative discussions as simple as possible, we consider N indistinguishable free fermions confined in a harmonic potential. The space $\mathcal{G}$ is the single-particle eigenenergy spectrum, i.e., the set of natural numbers $\mathbb{N}$. (The zero energy is irrelevant and ignored.) For a Fock state λ, $E = \sum_{\nu=1}^{\infty} \nu n_\nu$. This maps λ into a partition of integer E into N distinct summands, a research area for which Euler laid down a foundation [39]. To be precise, an eigenenergy ν, when its corresponding eigenstate is occupied ($n_\nu = 1$), mimics a summand. In 1941 the field of random integer partitions was opened up [40], and in the past few decades such partitions have been found to bear rich structures [25–27]. (One should not confuse this with the old subject of using standard statistical mechanics to study the number of partitions [41].) In particular, the observable $\varphi_\lambda(t) \equiv \int_t^\infty \sum_{\nu'=1}^\infty n_{\nu'}\delta(\nu' - \nu)d\nu$, counting at given λ the number of summands $\geqslant t$ or, equivalently, the number of particles occupying single-particle eigenstates with eigenenergies $\geqslant t$ [42], defines a random stepped curve. As proven rigorously [26], this curve has a limit shape. Our results have intimate relations to this.

For illustrations we consider the case where N is not fixed, i.e., $\mu = 0$. In this case for sufficiently large E, Eq. (10) shows that for an overwhelming number of λ,

$$\varphi_\lambda(t) \overset{E\gg1}{=} \int_t^\infty \frac{d\nu}{e^{\nu/T}+1} = T\ln(1+e^{-t/T}), \quad T = \frac{\sqrt{12E}}{\pi} \tag{17}$$

$[E = \int_0^\infty \varphi_\lambda(t)dt]$, in agreement with the theorem [26]

$$\lim_{E\to\infty} \mu^E\left\{\lambda : \left|\frac{1}{\sqrt{E}}\varphi_\lambda\left(\sqrt{E}t\right) + s(t)\right| < \epsilon\right\} = 1 \quad \forall\epsilon > 0.$$

Here, $s(t)$ is given by the Vershik curve, $e^{-\frac{\pi s}{\sqrt{12}}} - e^{-\frac{\pi t}{\sqrt{12}}} = 1$. This theorem implies that if the set of all partitions λ defined above is equipped with a uniform probability measure μ^E, then a typical partition has a limit shape of $-s$. Our method, though less rigorous, has the advantage of being applied to more general conditions and systems.

Indeed, for generic N, E, for which rigorous results are not available, we have confirmed Eq. (10) numerically. Specifically, we use the Monte Carlo method to draw randomly a partition $\lambda = \{n_\nu\}$ of E (with N distinct summands) from the uniform probability measure. As shown in Fig. 3, a typical λ (green dashed curve), though appearing to be random, has a nonrandom Λ (stepped curve) fitted well by $f_{\rm FD}$ (red dashed curve) [30]. The pattern of $\{\frac{N_m}{G_m}\}$ concentrates on the smooth curve $f_{\rm FD}$. (G_m corresponds to the scale over which C_ν varies and thus to $M_{rr'}$ with specific ranges of r, r'.) We see that $f_{\rm FD}$ is justified not only for an individual λ, but also for N as small as 400.

Summarizing, we have shown analytically how the exchange interaction, namely, the particle indistinguishability, gives rise to a hidden thermal structure in the Fock space, and opens up a door to the emergence of thermal equilibrium phenomena from eigenstates of many-body systems with or without the direct interaction between particles. Furthermore, we have uncovered a striking mechanism for eigenstate thermalization of a nonideal Fermi gas on a torus [Fig. 1(a1)]. It is natural to generalize this result to a nonideal Fermi gas in an Anderson localized cavity [Fig. 1(c)]. This issue is currently under investigation. We expect the outcomes to shed light on the many-body localization [43–45], especially in view of the fact that the many-body localization is equivalent to localization in the Fock space. In this Rapid Communication, we focused on the kinematic aspect. The dynamical aspect, especially the interplay between relaxation and the hidden thermal structure, is an important issue in future studies. In a separate work [46], we will show that given a typical nonthermal initial state, under the unitary evolution, the observable will relax to the thermal value discussed here, and the relaxation time is the Ehrenfest time.

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[31] In fact, for a quantum gas on a d-dimensional torus we have d more constraints, imposed by the conservation laws of the total momentum. However, these additional constraints are not essential for this system. Indeed, if we introduce d Lagrange multipliers, $(u_1, u_2, \dots, u_d) \equiv u$ corresponding to these constraints, the distribution (10) is modified to be $(e^{(p^2/2-u\cdot p-\mu)/T}+1)^{-1}$, which is similar to the generalized Gibbs ensemble [9]. This modified distribution can be rewritten as $(e^{[(p-u)^2/2-\tilde{\mu}]/T}+1)^{-1}$, where $\tilde{\mu}=\mu+\frac{1}{2}u^2$. From this we see that the only physical consequence introduced by the additional constraints is that the torus moves in a velocity u (with respect to the laboratory reference frame). However, we can always adjust the reference frame in which the torus is at rest.
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