Letter

Graviton chirality and topological order in the half-filled Landau level

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The fractional quantum Hall state at the Landau level filling factor 5/2 is extremely interesting because it is likely the first non-Abelian state, but its precise nature remains unclear after decades of study. We demonstrate this can be resolved by studying the chirality of its graviton excitations, using circularly polarized Raman scattering. We discuss the advantage of this bulk probe over the existing edge probes.

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Introduction and motivation. Non-Abelian fractional quantum Hall (FQH) liquids are arguably the most exotic quantum states of matter, which can provide a platform for topological quantum computation. The most promising candidate for such a liquid is the one at the Landau level (LL) filling factor $\nu = 5/2$ [1], and the leading candidate states (based on extensive numerical studies [2–15]) are the Moore-Read (MR) Pfaffian state [16], and its particle-hole conjugate partner, the anti-Pfaffian (APf) state [17,18], both describing electrons in a half-filled LL. In the absence of LL mixing and other symmetry-breaking perturbations, a half-filled LL possesses particle-hole symmetry, as a result of which the MR and APf states are exactly degenerate. LL mixing breaks particle-hole symmetry and favors the APf state [19–22]. The situation is much murkier on the experimental front. It has been long believed that the MR and APf states, while topologically distinct, can only be distinguished in their edge properties. As a result, existing experiments attempting to determine the nature of the 5/2 state have been focused on the edge (for a review of earlier experimental work that also includes bulk spin polarization measurements which are consistent with both MR and APf states, see Ref. [23]). Among them perhaps the most direct probe is the recent thermal Hall conductance measurement [24]. While the discovery of half-integer quantization definitely points to the non-Abelian nature of the 5/2 state, its specific value turns out to be consistent with neither the MR nor APf state, but suggests a particle-hole symmetric state instead. This (apparent) particle-hole symmetry could be due to the spatial mixture of MR and APf liquids in the sample, that form either spontaneously [25] or due to disorder that locally breaks the particle-hole symmetry [26–29], which could yield an edge structure that gives rise to the measured thermal Hall conductance. The viability of this scenario is currently under debate [30,31]. Another controversial explanation of the experiment is the lack of equilibration at the edge [32–34], which is an extrinsic effect. There is, of course, the possibility of an intrinsically particle-hole symmetric FQH state known as particle-hole Pfaffian (PH Pf) [35,36], but none of the numerical studies [37–39] have seen a clear gapped phase or a state that can energetically compete with either the MR or APf state [39] (see also Ref. [40]).

In this Letter we point out that the MR, APf, PH Pf (or any other intrinsically particle-hole symmetric FQH state), and in principle their spatial mixtures, can be distinguished by measuring the chirality of a *bulk* geometric excitation termed a graviton [41], which is accessible via polarized Raman scattering [41–43]. In our earlier work [41] we demonstrated that for electron states (such as those in the Laughlin sequence with $\nu = 1/m$) the gravitons carry spin -2, and pointed out their particle-hole conjugate states at $1 - \nu \neq \nu$ the chirality is reversed and gravitons carry spin $+2$ (see also Ref. [44]). This, however, leaves the situation ambiguous at the particle-hole symmetric filling factor of $\nu = 1/2 = 1 - \nu$. It has already been demonstrated [41] that the MR graviton carries spin -2. The APf graviton then must carry spin $+2$, while both chiralities should be present in a particle-hole symmetric state. Should there be a mixture among these different states, the local chirality can be revealed as long as the probing light can be localized in a region smaller than the domain size. In addition to the obvious and potentially far-reaching experimental relevance, our results also reveal the deep connection between the geometric [44–50] and topological [51] aspects of the FQH effect (which has been perhaps somewhat underappreciated thus far), and point to the possibility of *bulk* probes of topological order (for an earlier suggestion in this general direction, see Ref. [52]).

Models and graviton operators for the 5/2 state. As shown in Refs. [41,48], electrons in an LL couple to an external oscillating metric through a set of two-body graviton operators, whose spectral functions describe the absorption rate of the "gravitational wave" propagating through the system. The graviton operators we employ here are different from their lowest LL counterparts [41] and are modified by the presence of a nontrivial LL form factor, and can be derived the same way as in Ref. [48],

$$\hat{O}^{(2)}_{\pm}(n) = \sum_{q_x,q_y} (q_x \pm i q_y)^2 V(q) e^{-q^2/2} \bar{\rho}(\boldsymbol{q}) \bar{\rho}(-\boldsymbol{q}) F_n(q), \qquad (1)$$

$$F_n(q) = |L_n(q^2/2)|^2 - 2L_n(q^2/2)L'_n(q^2/2), \qquad (2)$$

where n is the LL index, $V(q)$ is the Fourier transform of the Coulomb potential, L_n is the nth Laguerre polynomial, the projected density operator is $\bar{\rho}(\boldsymbol{q}) = \sum_n e^{-i\boldsymbol{q}\cdot\boldsymbol{R}_n}$, and $\boldsymbol{R}$ is the guiding center coordinate. The prime on L_n signifies the derivative with respect to the argument. The wave vector q is measured in units of inverse magnetic length $1/\ell$, where $\ell = \hbar/eB$. $\hat{O}^{(2)}_{\pm}(n)$ describes coupling to the gravitational wave with opposite (circular) polarizations, that change the angular momentum of the electron liquid by ± 2, respectively.

The Hamiltonian for the Coulomb repulsion for the nth LL is

$$H_n = \frac{1}{2}\sum_{q_x,q_y} V(q)e^{-q^2/2}\bar{\rho}(\boldsymbol{q})\bar{\rho}(-\boldsymbol{q})f_n(q),$$

$$f_n(q) = L_n^2(q^2/2).$$

In this Letter we focus on the valence electrons at $\nu = 5/2 = 2 + 1/2$ that half fill the second LL with index $n = 1$. The form factors of the graviton operator and the Hamiltonian simplify to $F_1(q) = (1 - q^2/2)(3 - q^2/2)$ and $f_1(q) = (1 - q^2/2)^2$, respectively. In some cases we have also increased the first Haldane pseudopotential of the Coulomb repulsion by a small amount. It is important for our purposes to also break particle-hole symmetry by introducing a weak three-body interaction. The exact form is immaterial and we choose the simplest case for which the MR state is a zero energy ground state [53]. We will also use its attractive counterpart by flipping its sign. Such additional pseudopotential terms also contribute to the graviton operators, but do not involve LL form factors (see Ref. [41]).

In experiment, LL mixing breaks PH symmetry by generating a slew of three-body pseudopotentials from the two-body Coulomb repulsion, which have been calculated perturbatively [54,55] in the LL-mixing parameter $\kappa = \varepsilon/\hbar\omega$, where $\varepsilon = e^2/4\pi\epsilon\ell$ is the Coulomb interaction scale, and ϵ is the dielectric constant of the material. In most of what follows we quote energies in units of ε. We also set $\hbar = 1$ and ignore the width of the electron layer. For weak LL mixing the strongest component corresponds to the MR pseudopotential and is negative: -0.0147κ.

Numerical calculations. Our calculations are on high-symmetry tori, namely square and hexagonal geometries. These are somewhat complementary and are helpful in discerning finite-size effects. Below we review the known characteristics for both MR and APf model states (exact ground states of idealized three-body model Hamiltonians) as well as for generic states. For even numbers of electrons the topological sectors [excluding the twofold center-of-mass (c.m.) degeneracy] are either a triplet (hexagonal) or split into a doublet and a singlet for square symmetry. For the model Hamiltonians, all three ground states are degenerate with zero energy in any geometry. Only their respective crystal momenta are different for different geometries. In hexagonal geometry these are at the three corners of the Brillouin zone (BZ). In the case of the square unit cell the singlet is at the zone corner (ZC) (1,1), while the doublet is at the zone boundary (ZB) (0,1) (1,0). For generic states in the presence of PH symmetry and for even electrons, the K vectors of the topological sectors are the same as in the model states. The degeneracy, however, is different for square geometry. There is a small splitting of energy between the singlet and the doublet (ZB). Depending on size both the singlet and the doublet could become the absolute ground state. In our calculations we have assumed that both are valid candidates irrespective of which one is the absolute ground state. The splitting is a finite-size effect and the degeneracy is recovered for large sizes.

For the model Hamiltonians with odd numbers of electrons there is one zero energy ground state with $K = 0$ at the zone center, corresponding to the only topological sector for all geometries.

For the generic case, in hexagonal geometry and depending on whether the number of electrons modulo 6 is one or not, the ground state is a singlet or a doublet, respectively. Both topological sectors of the MR and APf are represented by the doublet [56]. This is an interesting case and we will return to discuss it later.

In all cases we calculate the spectral functions of the graviton operators [41],

$$I_{\pm}(\omega) = \sum_n |\langle\Psi_n|\hat{O}^{(2)}_{\pm}|\Psi_0\rangle|^2\delta(\omega - \omega_n), \tag{3}$$

where $|\Psi_0\rangle$ is a ground state, which is included in the sum over intermediate states. As a result, the total graviton weight can be normalized to one by dividing the right-hand side of the above by $\langle\Psi_{0\pm}|\Psi_{0\pm}\rangle$, where $|\Psi_{0\pm}\rangle = \hat{O}^{(2)}_{\pm}|\Psi_0\rangle$, so that $\int I_{\pm}(\omega)d\omega = 1$.

Square geometry. In this geometry for an even number of electrons and the ground state doublet there is a conserved unitary operator that results from the product of two antiunitary mirror and PH conjugation operators. The entire energy spectrum can be classified by a Z_2 parity quantum number. However, for ZB (ground and excited) states the chiral graviton operator has mixed parity. That is, the real and the imaginary parts of $\hat{O}^{(2)}_{\pm}$ produce states with opposite parities. This means the two parts are not present simultaneously and hence the graviton weight is always nonzero. A finite graviton weight for the ground states, however, is an undesirable effect and will be removed below.

In contrast, for the singlet ground state as well as the excited states, there are angular momentum selection rules irrespective of the presence or absence of PH symmetry. Some states have a finite graviton weight and some not, according to whether their angular momentum is within ± 2 of the ground state. However, when PH symmetry is broken the weights are different for the chiralities ± 2, but they occur for the same states because on a square the discrete angular momentum $2 = -2 \mod (4)$.

Since the energies and the graviton weights are identical for the case of degenerate ground states, we include them in the intermediate sum of Eq. (3) and trace over the ground states. It proves convenient to combine the two ZB ground states as follows: $|\Psi_0\rangle_{\pm} = \frac{|\psi_0\rangle_1 \pm |\psi_0\rangle_2}{\sqrt{2}}$. The wave functions in the two (1,2) sectors have different translational quantum numbers and are orthogonal. The graviton operator preserves these quantum numbers and hence the sums of matrix elements over the excited states are now included for both sectors. The contribution of the ground state to the sum is the square of ${}_1\langle\psi_0|\hat{O}^{(2)}_{\pm}|\psi_0\rangle_1 + {}_2\langle\psi_0|\hat{O}^{(2)}_{\pm}|\psi_0\rangle_2 = 0$, thus dropping out

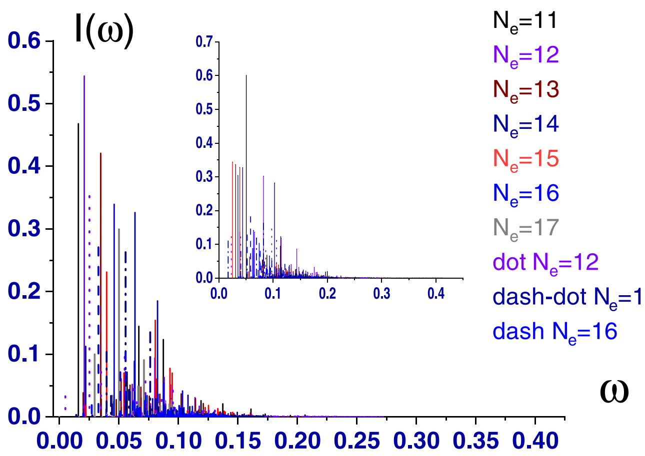

FIG. 1. Graviton spectral functions for 11–17 electrons on a square unit cell. For an even number of electrons we have included data for both ZC (see text) and ZB (dotted lines). Because of PH symmetry the spectra for positive and negative chirality are identical. In the inset we have added a $v_1 = 0.035$ Haldane pseudopotential to the $n = 1$ Coulomb interaction. The overlaps with MR or APf states are at or near their maximum for this v_1.

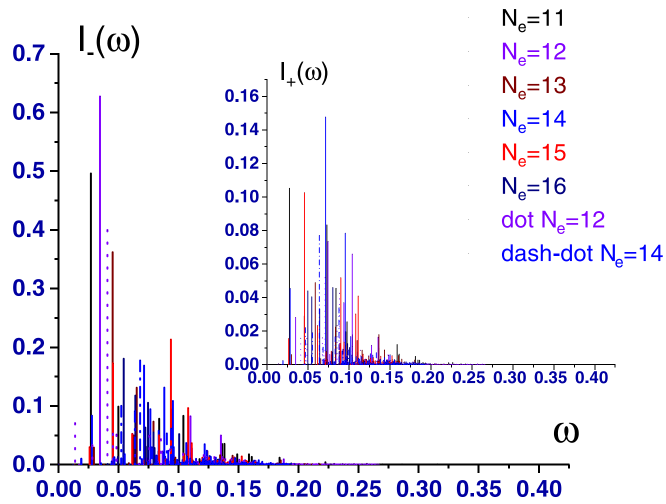

FIG. 3. Same as in Fig. 2 except we have added a three-body pseudopotential with the opposite sign (0.01). The stronger intensity is for -2 chirality. The inset shows the result for $+2$ chirality, which is suppressed.

as verified numerically (to machine precision) for all cases that we have studied. This removes the graviton weight of the ground state, which is always absent for an odd number of electrons, because of angular momentum selection rules.

We start with the case of a pure Coulomb interaction. The PH symmetry is present in this case, and the ground state can be viewed as the PH-symmetrized MR state [3]. As a result, we have $I_+(\omega) = I_-(\omega)$, which is presented in Fig. 1. Similar to the cases studied in Ref. [41], we observe fairly sharp peaks indicating the presence of graviton excitations in the system, except they come with both chiralities. In Figs. 2 and 3 we show the graviton spectral functions in the presence of small three-body PH symmetry-breaking interactions. In calculating the relative weights of two chiralities we normalize the weaker spectrum by the total weight of the stronger.

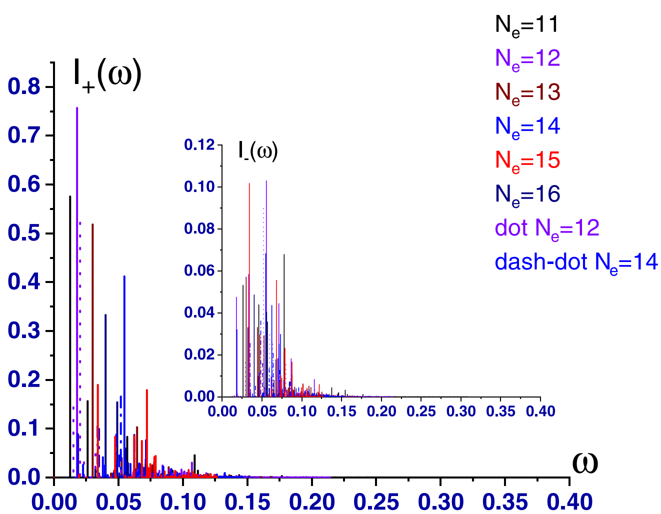

FIG. 2. Same as in Fig. 1 except that we break PH symmetry by introducing a three-body interaction, described in the text, with a strength of -0.01. The dominant spectrum is for the $+2$ chirality. The inset shows the spectrum for -2 chirality, and the response is seen to be suppressed by nearly an order of magnitude.

Figure 2 corresponds (roughly) to the case of LL-mixing parameter $\kappa \approx 0.7$, which is representative of realistic situations, and tilts the ground state toward APf. While this results in a very small negative three-body potential, it has a dramatic effect on the spectral functions: We find I_+ dominates I_-, with the total weight of the latter reduced to about 20% of the former. This indicates gravitons with angular momentum $+2$ dominate the gravitational response of the system, which is in a holelike APf state. In Fig. 3 we reverse the sign of the three-body potential which favors the Pfaffian state, and the situation is reversed: I_- dominates I_+, with the total weight of the latter reduced to about 30% of the former. This indicates gravitons with angular momentum -2 dominate the gravitational response of the system, as we already saw in Ref. [41] for the Pfaffian state. We thus find the graviton chirality is opposite for the Pfaffian and APf states, and can be used to distinguish them experimentally (more on this point later). It is important to note that while disorder breaks rotation symmetry and may mix states with different angular momenta, chirality is a more robust property of the system which remains well defined when particle-hole symmetry is broken.

Hexagonal geometry. Here, for an even number of electrons, the topological sector is a set of threefold degenerate (related by rotations) ground states and the symmetry analysis of the graviton operator and the ground states is more complicated. Notwithstanding, the ground state weight can be removed by a set of new orthogonal states, as in the case of ZB doublets, $|\Psi_0\rangle_a = \frac{\alpha|\psi_0\rangle_1+\beta|\psi_0\rangle_2+\gamma|\psi_0\rangle_3}{\sqrt{3}}$, where $\alpha = e^{2i\pi/3}$, $\beta = e^{4i\pi/3}$, and $\gamma = -\alpha - \beta = 1$ are the cube roots of unity. The other two states $|\Psi_0\rangle_b$ and $|\Psi_0\rangle_c$ are obtained by cyclic permutations of α, β, and γ. Again, the three expectation

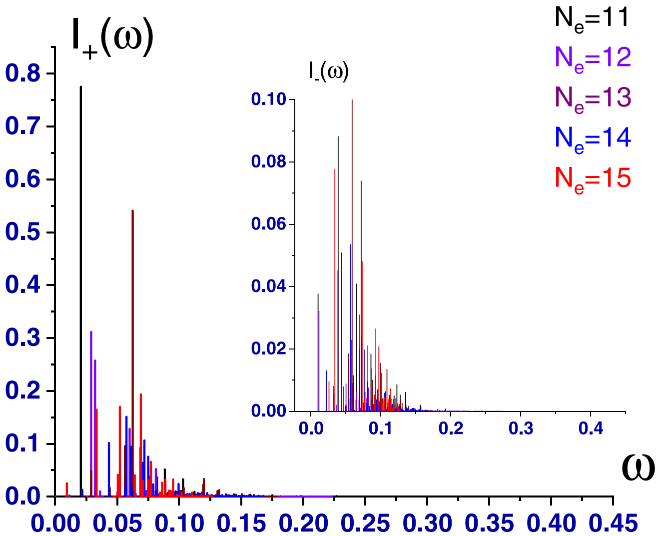

FIG. 4. Graviton spectral functions for 11–15 electrons on hexagonal geometry in the presence of a three-body potential of strength −0.01. As in the case of square geometry, the +2 chirality is dominant while the −2 chirality (inset) is more strongly suppressed than in the inset of Fig. 5.

values of $\hat{O}^{(2)}_{\pm}$ add to zero for all three ground states:

$$\alpha({}_1\langle\psi_0|\hat{O}^{(2)}_{\pm}|\psi_0\rangle_1) + \beta({}_2\langle\psi_0|\hat{O}^{(2)}_{\pm}|\psi_0\rangle_2) + \gamma({}_3\langle|\hat{O}^{(2)}_{\pm}|\psi_0\rangle_3) = 0.$$

Figures 4 and 5 are the hexagonal counterparts of Figs. 2 and 3, where we see very similar behavior. The consistency between different geometries is an indication that finite-size effects are overall minimal in our calculations.

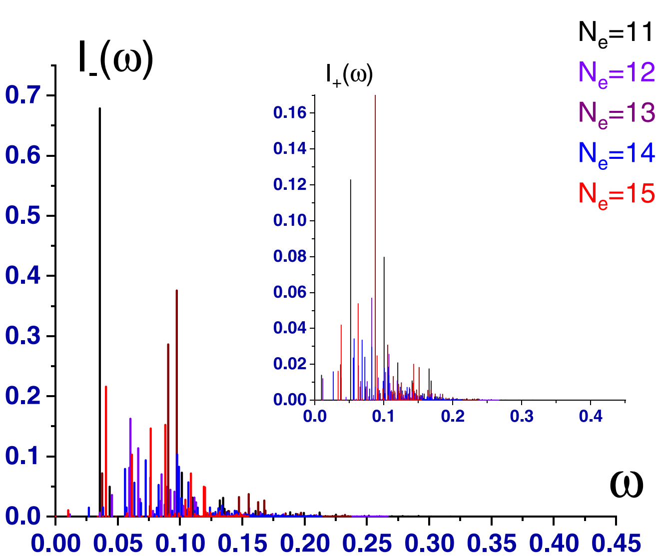

FIG. 5. Same as Fig. 4 except for a repulsive (0.01) three-body potential. The suppression of the opposite chirality (inset) is somewhat less suppressed than in the Fig. 4 chirality. The inset shows the spectrum for positive chirality, and the response here is also suppressed.

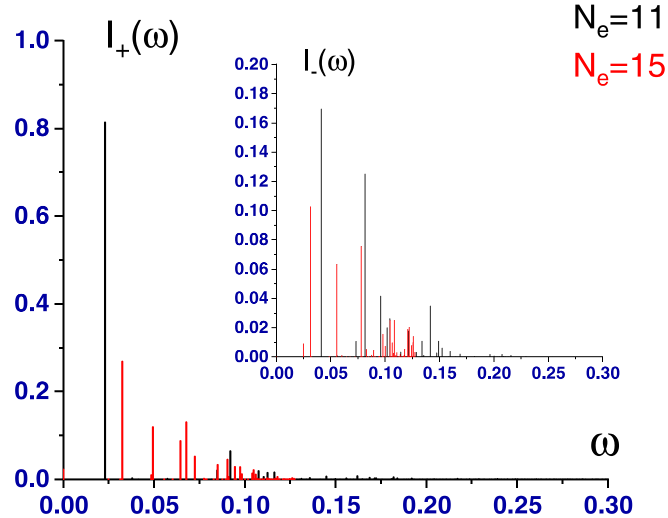

FIG. 6. The graviton spectrum for doublet ground states that can occur for odd electrons. We have added a three-body $M = 3$ pseudopotential -10^{-6}. The red line at zero comes from the doublet member which is split off by the weak potential. The inset shows the spectrum for negative chirality. It is suppressed by a factor of ≈5.0. The added three-body potential has little role in breaking PH symmetry.

We now return to the case of generic interactions for an odd number of particles. The ground state becomes a degenerate doublet resulting from the combined presence of antiunitary PH symmetry and sixfold discrete rotation symmetry [56,57]. The doublets appear for sizes when $N_e \mod (6) \neq 1$. Similarly, in these cases the MR state and the APf are orthogonal [56]. We find that our code already breaks the PH symmetry spontaneously. However, because of the degeneracy the two different angular momenta are mixed and the graviton weights become nonzero for every state. The addition of a very small three-body potential (of magnitude 10^{-6}) lifts the degeneracy and restores the angular momentum selection rules.

Figure 6 shows that $I_+(\omega)$ is dominant while $I_-(\omega)$ is suppressed. If the sign of the three-body pseudopotential is reversed, then the plot looks the same, except I_+ and I_- are exchanged.

On the torus in some cases the MR and APf have nonzero overlap. This in part may be responsible for some of the larger peaks in suppressed chiralities shown in the insets. For example, for $N_e = 14$ (solid line in Fig. 3) the overlap is about 47%. The graviton appears to reflect the "dual" nature of the ground state. A similar trend may explain the large peak for $N_e = 13$ (Fig. 5) with hexagonal geometry where the overlap is about 42%. These are finite-size effects, which will not occur in experiment. In fact, the graviton is a sensitive probe; it can detect the phase changes seen in the wave function under LL mixing. The first of such transitions occurs at an $M = 6$ three-body pseudopotential from APf to the MR phase. We have verified that the graviton not only detects this phase but shows it to be a tepid one. The system reverts back to APf when an $M = 9$ psudopotential is included [21,58].

Discussion and summary. We have calculated graviton spectral functions for Hamiltonians appropriate for the $\nu = 5/2$ FQH state. While originally formulated as the system's response to a "gravitational wave" [41,48], it was anticipated that the gravitons and in particular their chiralities are detectable experimentally by Raman scattering of circularly polarized light [41,42].

In an important recent paper [43], Nguyen and Son demonstrated *explicitly* that the Raman spectral functions are *identical* to the graviton spectral functions calculated here and in Ref. [41], if the small anisotropy of the valence band is neglected. This makes it possible to directly probe the gravitational response and, in particular, detect quantum gravitons using this fairly standard experimental probe. It also facilitates a *quantitative* comparison between our theoretical results (both here and those of Ref. [41]) and experiment. We note Pinczuck and co-workers' earlier results on the $1/3$ Laughlin state [59] are in good agreement with our calculations [41], although the graviton chirality could not be extracted since they used unpolarized light.

In sharp contrast to the $1/3$ state, the situation is much murkier at $5/2$, with many competing theoretical proposals. We demonstrated the leading candidates based on numerics, Moore-Read Pfaffian and anti-Pfaffian, can be clearly distinguished by the chiralities (∓ 2, respectively) of their graviton excitations, which are detectable using circularly polarized Raman scattering. We emphasize this is a *bulk* probe which does *not* suffer from many complications and subtleties at the edge. We note recent thermal transport experiments favor a particle-hole symmetric state at $5/2$ [24,60]. This could be due to the presence of domains of Pfaffian and anti-Pfaffian states in the system [25–28,31]. Such domains can also be revealed by Raman scattering, as long as their sizes are larger than the spatial resolution of the experiment. While we do not have a microscopic model that stabilizes an intrinsically particle-hole symmetric state, as discussed earlier we expect on general grounds that gravitons with both chiralities should be present and contribute (roughly equally) to the Raman scattering intensity of light with both circular polarization. We thus conclude polarized Raman scattering can potentially resolve all of the leading candidates for the $5/2$ state.

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