Multiple Magnetorotons and Spectral Sum Rules in Fractional Quantum Hall Systems

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We study, numerically, the charge neutral excitations (magnetorotons) in fractional quantum Hall systems, concentrating on the two Jain states near quarter filling, $\nu = 2/7$ and $\nu = 2/9$, and the $\nu = 1/4$ Fermi-liquid state itself. In contrast to the $\nu = 1/3$ states and the Jain states near half filling, on each of the two Jain states $\nu = 2/7$ and $\nu = 2/9$ the graviton spectral densities show two, instead of one, magnetoroton peaks. The magnetorotons have spin 2 and have opposite chiralities in the $\nu = 2/7$ state and the same chirality in the $\nu = 2/9$ state. We also provide a numerical verification of a sum rule relating the guiding center spin $\bar{s}$ with the spectral densities of the stress tensor.

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Introduction.—The exploration of topological phases of matter started with the discovery of the fractional quantum Hall (FQH) effect [1–3]. Under a strong magnetic field, electrons in two dimensions form strongly correlated quantum Hall systems. In the lowest Landau level (LLL) limit, the kinetic energy of the electrons is constant, and the two-dimensional electron system is driven to numerous exotic topological phases depending on the filling fraction and the effective interactions. The topological characteristics of the FQH ground state and its charged excitations can be understood using the wave function approach, pioneered by Laughlin [2] and further developed by many other authors [4–8]. The idea of the composite fermion (CF) by Jain [9,10] provides an explanation of numerous gapped FQH states observed in experiment [11,12] and an intuitive construction of model wave functions, and suggests the Abelian and non-Abelian braiding statistic of quasiparticles and quasiholes [13,14]. Inspired by Jain's intuitive picture, a field theory description of FQH states was developed based on the idea of flux attachment [15]. Subsequently, Halperin, Lee, and Read (HLR) proposed the theory for FQH state at half filling, which predicts a gapless Fermi-liquid state [16], that has been confirmed experimentally [17].

Recently, modifications to the HLR theory have been suggested to make it consistent with the symmetries of a single Landau level. Particle-hole symmetry, which has been long an issue [18,19], is restored in the Dirac CF theory for FQH states near half filling [20]. The dipole coupling of the CF to the electric field takes care of the consistency of the theory with diffeomorphism [21,22]. In combination, these two modifications lead to response functions consistent with all known symmetries (see, e.g., Ref. [23]).

An important feature of gapped quantum Hall states is the existence of the neutral magnetoroton mode, first suggested by Girvin, MacDonald, and Platzman (GMP) [24] and later observed in experiment [25–27]. Though GMP originally introduced the magnetoroton as a charge density wave, recent works [28–30], employing the lowest Landau level symmetries, suggest that the magnetoroton has spin 2 and thus can be considered as a massive "emergent graviton" in FQH systems. The magnetoroton has been studied from many different perspectives: by constructing the wave function [31,32] (which conforms with the spin-2 structure), as an excitation of the CF crossing Λ levels [33], by exact diagonalization [34–39], or within the Dirac CF theory [21,30,40], where it is interpreted as the shear deformation of the composite Fermi surface. In the latter studies, the chirality of the magnetoroton is determined by the direction of the residual magnetic field seen by the Dirac CFs, and the positions of the minima of the magnetoroton dispersion relation match the experimental results [25–27] rather well.

Very recently, the Dirac CF theory has been generalized to Jain states near $\nu = \frac{1}{4}$ [41–43]. The situation with the magnetoroton there seems to be very different from that near $\nu = \frac{1}{2}$: it is necessary [43] to postulate a "Haldane mode," i.e., an extra high-energy magnetoroton (or multiple magnetorotons), in addition to the low-energy magnetoroton that emerges from the dynamics of the CFs. The additional magnetoroton(s), which contribute to the projected static structure factor (PSSF), are crucial for the Haldane bound [44]

$$S_4 \geq \frac{|\bar{s}|}{4}, \tag{1}$$

where S_4 is the coefficient of the leading Q^4 [45] term in the PSSF, and $\bar{s}$ is the guiding center spin. On the LLL $\bar{s} = \frac{1}{2}(\mathcal{S} - 1)$ where $\mathcal{S}$ is the Wen-Zee shift. The extra magnetoroton was heuristically suggested to arise from the microscopic structure of the CF. While the electric dipole moment of the CF is constrained by its momentum [21,43], a higher moment deformation of its shape could in principle generate a spin-2 mode, which is the high-energy magnetoroton.

In this Letter we investigate the magnetoroton excitations, guided by the FQH spectral sum rules [43,46,47] relating the chiral graviton spectral functions with $\bar{s}$ and S_4. These sum rules constrain the spectral densities of the $\nu = p/(2np \pm 1)$ states and the Fermi-liquid-like state at $\nu = 1/4$, and suggest the chiralities of the magnetorotons there. We calculate the spectral densities numerically, from which we read out the chirality of the magnetorotons and verify the sum rules.

Graviton spectral sum rules.—In the LLL limit, when the interacting energy is much smaller than the cyclotron energy, one can obtain the exact sum rules involving the spectral densities of the stress tensor [46,47]. In the complex coordinate $z = x + iy$, the two components of the traceless part of the stress tensor, $T_{zz} = \frac{1}{4}(T_{xx} - T_{yy} - 2iT_{xy})$ and $T_{\bar{z}\bar{z}} = \frac{1}{4}(T_{xx} - T_{yy} + 2iT_{xy})$, can be used to define two spectral densities [46]

$$I_-(\omega) = \frac{1}{N_e}\sum_n |\langle n| \int d\mathbf{x}\, T_{zz}|0\rangle|^2 \delta(\omega - E_n), \tag{2}$$

$$I_+(\omega) = \frac{1}{N_e}\sum_n |\langle n| \int d\mathbf{x}\, T_{\bar{z}\bar{z}}|0\rangle|^2 \delta(\omega - E_n), \tag{3}$$

where N_e is the total number of electrons, $|0\rangle$ is the ground state, the sum is taken over all excited states $|n\rangle$ in the lowest Landau level, and E_n is the energy of the state $|n\rangle$ relative to the ground state. Physically, (2) and (3) are the densities of spin-2 states with opposite chiralities at frequency ω, and as such they depend on the microscopic details of the FQH problem. The expressions for the integrals of T_{zz} and $T_{\bar{z}\bar{z}}$ over space in terms of the LLL operators have been derived in Ref. [48]. We expect $I_-(\omega)$ and $I_+(\omega)$ to vanish at frequencies below the energy gap; we also expect them to rapidly go to 0 at energies much larger than the energy scale set by the Coulomb interaction. Using the $U(1)$ charge conservation and the LLL limit of momentum conservation, one can obtain the following exact sum rules [43,46,47,49,50]:

$$\int_0^\infty \frac{d\omega}{\omega^2}[I_-(\omega) + I_+(\omega)] = S_4, \tag{4}$$

$$\int_0^\infty \frac{d\omega}{\omega^2}[I_-(\omega) - I_+(\omega)] = \frac{\bar{s}}{4}\left(= \frac{\mathcal{S} - 1}{8}\ \text{on LLL}\right). \tag{5}$$

For derivations of the sum rules see Refs. [43,47,55]. Both sum rules do not rely on microscopic details and can be applied for fractional quantum Hall states in the single Landau level limit, where Landau-level mixing is ignored. By definition $I_\pm(\omega)$ are non-negative, therefore the sum rules imply the Haldane bound (1). This bound is saturated if and only if the FQH state is chiral, i.e., when one of the spectral densities vanishes identically [i.e., $I_-(\omega) = 0$ or $I_+(\omega) = 0$] [56].

General Jain states.—The Wen-Zee shift $\mathcal{S}$ of the general Jain state has been found previously [43,57,58]

$$\nu_+ = \frac{p}{2np + 1}, \qquad \mathcal{S}_+ = p + 2n, \tag{6}$$

$$\nu_- = \frac{p}{2np - 1}, \qquad \mathcal{S}_- = -p + 2n. \tag{7}$$

The subscript index $+$ ($-$) corresponds to the residual magnetic field seen by CFs being in the same (opposite) direction as the applied magnetic field [21,43]. The direction of the residual magnetic field determines the chirality of the low-energy magnetoroton, the one induced by the deformation of composite Fermi surface. If the residual magnetic field is in the same (opposite) direction of the external field, the low-energy magnetoroton has negative (positive) chirality. Consequently, we expect a low energy peak in $I_-(\omega)$ ($I_+(\omega)$) of state ν_+ (ν_-).

In Ref. [43] a Dirac CF model of the $\nu_\pm$ states was presented. The model is supposed to be reliable at large p and its result for the spectral densities can be summarized as

$$\nu_+:\ \frac{I_-(\omega)}{\omega^2} = \frac{p+1}{8}\delta(\omega - \omega_L) + \frac{n-1}{4}\delta(\omega - \omega_H), \qquad I_+(\omega) = 0, \tag{8}$$

$$\nu_-:\ \frac{I_-(\omega)}{\omega^2} = \frac{n-1}{4}\delta(\omega - \omega_H), \qquad \frac{I_+(\omega)}{\omega^2} = \frac{p-1}{8}\delta(\omega - \omega_L), \tag{9}$$

where ω_L and ω_H are the energies of the low- and high-energy magnetorotons, respectively. Note that the delta function $\delta(\omega - \omega_H)$ may be broadened by the decay of the high-energy magnetoroton or splits into several peaks. One can introduce the integrated spectral densities

$$\mathcal{I}_\mp = \int_0^\infty \frac{d\omega}{\omega^2} I_\mp(\omega). \tag{10}$$

The prediction of Ref. [43] reads

$$\nu_+ : \mathcal{I}_- = \frac{p+2n-1}{8}, \qquad \mathcal{I}_+ = 0, \tag{11}$$

$$\nu_- : \mathcal{I}_- = \frac{n-1}{4}, \qquad \mathcal{I}_+ = \frac{p-1}{8}, \tag{12}$$

and from the sum rule (5) one finds the S_4 coefficient of the $\nu_\pm$ states:

$$S_4(\nu_+) = \frac{p+2n-1}{8}, \qquad S_4(\nu_-) = \frac{p+2n-3}{8}. \tag{13}$$

Some remarks are in order. (i) For $n = 1$ (near half filling), there is no high-energy magnetoroton, and both $\nu_\pm$ states are chiral. (ii) For $n \neq 1$, only the ν_+ state is chiral, with S_4 saturating the Haldane bound, while the ν_- is not chiral. (iii) Strictly speaking, the formulas presented above are obtained in the large p limit, so the application of these formulas for the case of, say, $p = 2$ should be taken with a grain of salt. On the other hand, one may expect that the qualitative statements about the chirality of the magnetoroton modes are robust.

The Fermi-liquid states.—In the Fermi-liquid state with $\nu = 1/2n$ the CFs are in zero emergent magnetic field and form a Fermi liquid, whose excitations do not contribute to the sum rule (5). The only contribution to the sum rule (5) is from the high-energy magnetoroton (the Haldane mode). Thus we find

$$\mathcal{I}_- - \mathcal{I}_+ = \frac{n-1}{4}. \tag{14}$$

Note that since the state is ungapped, the notion of S_4 does not apply. Naively we can associate $n-1$ with the guiding center spin for of the Fermi-liquid state. This, in turn, can be explained if one thinks of the CF at $\nu = 1/2n$ as a CF at $\nu = 1/2$ state with $2(n-1)$ flux quanta attached. The CF at $\nu = 1/2$ has no spin, while each attached flux quanta increases the spin by $\frac{1}{2}$.

Looking at Eqs. (8), (9), and (14), we notice that the contribution of the Haldane mode to the guiding center spin of states near $1/2n$ is universal. In the Fermi-liquid state $\nu = 1/2$, the spectral densities $I_-(\omega)$ and $I_+(\omega)$ should be identical due to the particle-hole symmetry, therefore $\mathcal{I}_- - \mathcal{I}_+ = 0$. (The same should be valid for the PH-Pfaffian state [20].)

Numerical results.—The graviton spectral function has been investigated on both boson and fermion fractional quantum Hall effect (FQHE) states which include Moore-Read states [59] as well as Laughlin states in Refs. [60,61]. In this Letter we present some results on the guiding center spin $\bar{s}$, the coefficient S_4 of the Q^4 term in the PSSF, and show the FQHE graviton spectral functions.

We find good agreement with the theoretical predictions of Eqs. (4) and (5). We will present some of the results in this Letter, delegating to the Supplement Material some others not germane to the main topic of the Letter. To obtain the correct sum rules for the Coulomb interactions it is necessary to use the complete stress tensor given in the Supplemental Material (SM) [51] to evaluate the spectral densities $I_-(\omega)$ and $I_+(\omega)$.

In Figs. 1 and 2 we present the results for the sum rules for the Jain states at fillings 2/5, 2/7, and the Fermi liquid state 1/4. We use the left vertical axis to represent $\bar{s}/4$, while the right vertical axis represents S_4. As predicted by Haldane [44] only chiral states saturate the S_4 bound. Generic states such as the ones obtained from the Coulomb interaction exceed this bound [43,46]. The numerical results of $\bar{s}$ and S_4 of Jain states converge to the theoretical predictions in Eqs. (13), (6), and (7). We also numerically

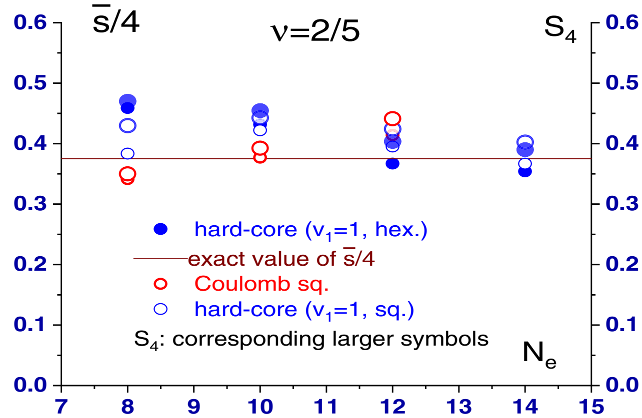

FIG. 1. Graphical representation of $\bar{s}$ and S_4 obtained from the sum rules at electron filling factor of 2/5 for both the Coulomb and the hard-core potentials. The latter is just a Haldane pseudopotential with its value set to 1. We see that $\bar{s}/4$ and S_4 are mostly identical, which is expected for a chiral state.

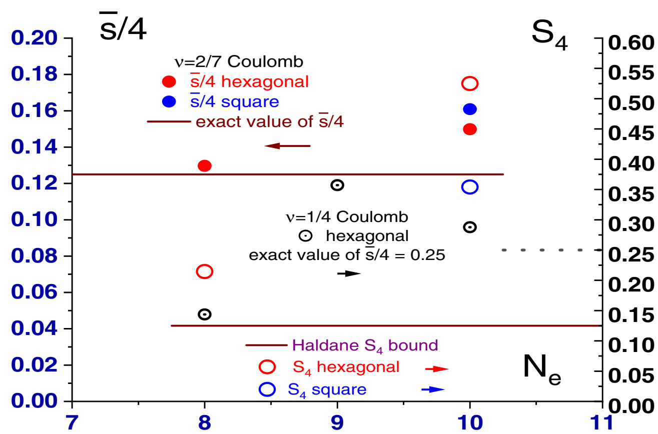

FIG. 2. Same as in Fig. 1 but at electron filling factors $\nu = 2/7$ and $\nu = 1/4$ for the Coulomb potential. The results for the 2/7 for $\bar{s}$ are given by the filled symbols and their values are on the left vertical axis (LVA). The RVA gives results for the S_4 of 2/7 (open symbols) and for $\bar{s}/4$ of $\nu = 1/4$ (open symbols with a central dot). The long lines show the exact values of $\bar{s}/4$ on both vertical axes. The short dotted line gives the same but for 1/4 state.

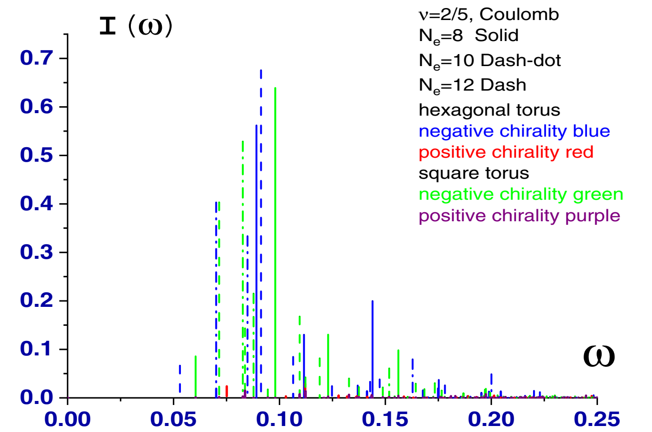

FIG. 3. The graviton spectral function for the filling factor $\nu = 2/5$ and pure Coulomb potential. The positive chirality total strength of the spectral function is much less than the negative chirality. As a result we have normalized the spectrum of both chiralities by the total weight of the gravitons with negative chirality. The quasichiral nature can be attributed to the dearth of quasiholes of the chiral parent, which is $1/3$ for the hierarchy [62]. Similarly, in Jain's CF approach, the spectrum is nearly chiral if there is not any opposing flux (to the attached ones) from the filled LLs.

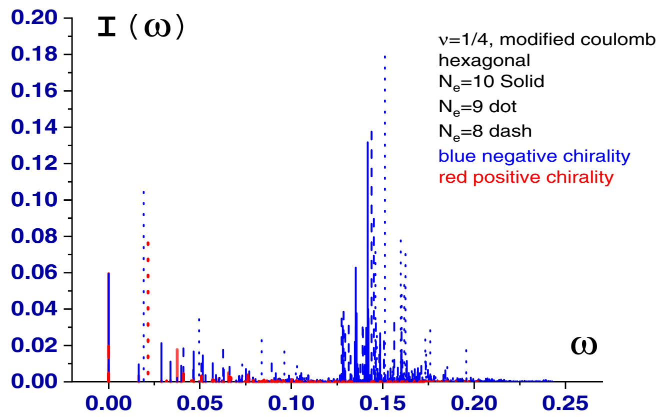

FIG. 5. The spectral function but for $\nu = 1/4$ composite Fermi liquid for the modified Coulomb potential. Note that the low energy spectrum has an equal weight for both chiralities with our normalization. This means the contribution to $\bar{s}$ is solely due to the Haldane mode.

verify the Wen-Zee shift of the Fermi liquid state $\nu = 1/4$ predicted by Eq. (14). The numerical results presented in Figs. 1 and 2 are highly nontrivial, they are the first numerical check of the exact graviton spectral sum rules in the LLL limit.

Figures 3–6 present the FQH graviton spectral functions for $\nu = 2/5$, $2/7$, $1/4$, and $2/9$. For $\nu = 2/5$, we use just the Coulomb potential since we want to compare it to the spectrum obtained with the hard-core potential, but for the remaining three fractions we calculate the spectral functions using a modified version of Coulomb. The two forms give the same qualitative results for the distributions of the graviton weights, particularly in instances where there are two distinct energy sectors.

All the theoretical predictions the graviton's chirality for the general Jain states from Dirac CF model of Ref. [43] are confirmed numerically. For $n = 1$, the chirality of gravitons is determined by the residue magnetic field seen by the CFs, therefore the graviton of $\nu = 2/5$ has negative chirality as showed in Fig. 3. With $n = 2$ the chirality of the low energy graviton is also determined by the residue

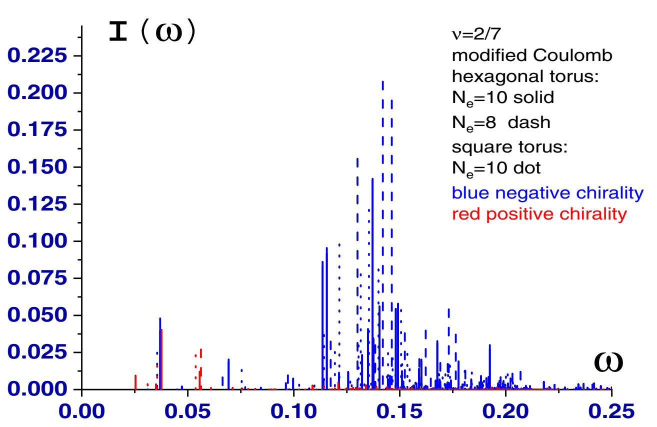

FIG. 4. Same as in Fig. 3 but for $\nu = 2/7$ filling and with the modified Coulomb. A clear separation of predominantly positive chirality is seen on the left. On the right is the spectrum for the negative chirality, which appears to contain the main weight of total spectral density. Here we have the opposite case from $2/5$. Hierarchy involves quasiholes and filled LLs in the Jain construction have an opposite B field to the attached fluxes.

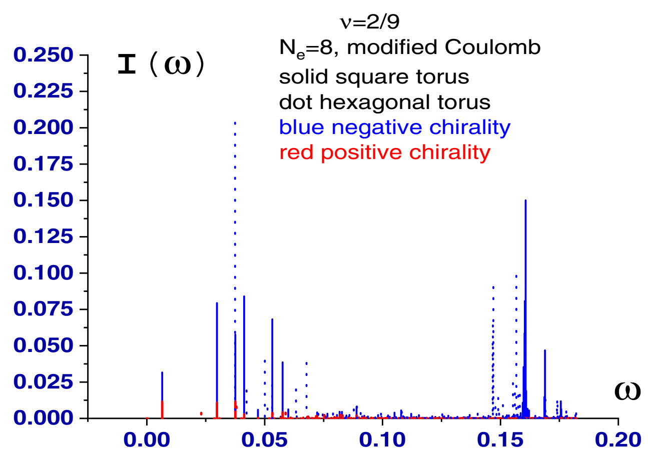

FIG. 6. The spectral function but for $\nu = 2/9$ with the modified Coulomb potential. The spectrum is nearly completely chiral as would be expected since it is to the right of the $1/5$ in the hierarchy or involves filled LLs with no opposing flux in the Jain's construction as in $2/5$ filling. Two well separated sectors with negative chirality can be seen clearly. Here the size is too small for us to get a meaningful $\bar{s}$ and S_4. However, the qualitative feature of having two separated regions of nontrivial graviton spectral density will persist for larger sizes.

magnetic field, and the chirality of the high energy graviton is universal for all Jain states near $1/4$. The predictions are confirmed in Figs. 4 and 6 [63].

Interestingly, the graviton spectral functions of Fermi-liquid state $\nu = 1/4$ in Fig. 5 shows the high energy graviton with expected chirality. From both the shift sum rule and the spectral densities, we see that the Haldane mode is universal for all FQH states near $1/4$: it does not care if there is a composite Fermi surface or if there is a residue magnetic field. Figure 5 also shows the low-energy excitations of both chirality with equal weight. As shown in the Supplemental Material [51], the spectral densities of the Fermi-liquid state $\nu = 1/2$ share the same feature with the low energy spectral densities of $\nu = 1/4$ with $I_-(\omega)$ and $I_+(\omega)$ being similar, the only difference is the absence of the Haldane mode.

Conclusion.—In this Letter, we compute numerically the graviton spectral densities for the $\nu = 2/7$, $2/9$, and $1/4$ states. For the first two states, we observe in the spectral densities two magnetoroton peaks, one at low energy with chirality that depends on the residual magnetic field acting on the CFs (and thus are opposite for the $\nu = 2/7$ and $2/9$ states), and one at high energy with the same chirality for the two states. The higher-energy magnetoroton is also observed in a spectral density of the $\nu = 1/4$ state, and this magnetoroton has approximately the same energy in all three filling fractions. The result is consistent with the two-magnetoroton model of FQH states near $1/4$ proposed in Ref. [43]. In addition, we have verified the FQH graviton spectral sum rules for the two Jain states and the Fermi liquid state $\nu = 1/4$.

We hope that our results will motivate the experimental exploration of the magnetoroton spectrum of the FQH states near $\nu = 1/4$, as well as more detailed study of the Haldane mode which may reveal its nature. With the guidance of the sum rules, one can use the numerical tool developed in this Letter to investigate various FQH states, both fermionic and bosonic.

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