Letter

Microscopic theory of field-tuned topological transitions in the Kitaev honeycomb model

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We microscopically construct an Abelian mutual Chern-Simons lattice gauge theory for magnetic field-tuned topological transitions in the Kitaev model and obtain a complete characterization of the phases, including their quasiparticles. At low fields for both ferro (FM) and antiferromagnetic (AFM) Kitaev interactions, we demonstrate non-Abelian Ising topological order (ITO), and explicitly construct the Majorana anyon as an intrinsic excitation—a twist defect in our *Abelian* gauge theory. For the AFM case, an Abelian chiral phase appears at intermediate fields with trivial topological order and fermionic bulk excitations. Remarkably, both the ITO phase and the intermediate phase have the same chiral central charge $c = 1/2$, implying no change in the quantized thermal Hall response across the transition. For the FM case, there is a direct transition from ITO to a partially polarized nontopological phase. Our paper completes the proof of Kitaev's original proposal of the low-field ITO, with $c = 1/2$ going beyond his mean-field arguments by including the crucial effect of the gauge fluctuations, and provides a resolution of the debate surrounding the intermediate field phase in the AFM case.

Study of topological orders and their stability against perturbations is a central topic of current interest in condensed matter physics [1–13]. It was originally proposed by Kitaev [12], using a mapping to a mean-field chiral p-wave superconductor (pSC) model, that the Zeeman-perturbed isotropic Kitaev model has non-Abelian Ising topological order (ITO) because (a) Majorana zero modes associated with *extrinsic* twist defects (π-flux particles) of the pSC obeyed Ising braiding rules and (b) the chiral central charge of the mean field model was $c = 1/2$. However, the effect of the gauge sector must be included to demonstrate the non-Abelian Majorana anyon as an *intrinsic* bulk excitation, and also obtain the correct values of c, topological entanglement entropy γ, and ground state degeneracy (GSD). For mean field pSC models, it is long known that three or four different ground states on the torus are possible [14]; however, such ground states are unique and do not simultaneously belong to a degenerate ground state sector. Specifically, the threefold degenerate ITO state in this case is known to require non-Abelian gauge fluctuations [14], while in Kitaev's fermionization scheme, the gauge fields that appear naturally are Abelian Z_2. In the low-field regime, recent perturbative treatments [15,16] emphasize the important role of vison (gauge field) fluctuation effects, and numerical studies suggest a low-field ITO phase [15–19] but the understanding of quasiparticles, necessary for complete characterization of the topological phases, is lacking. In the antiferromagnetic (AFM) case, there is an ongoing debate regarding the nature of the intermediate field phase

[17,19–29]. The high field phase is topologically trivial and partially polarized. For the ferromagnetic (FM) Kitaev model, there is a direct field-tuned transition from the Kitaev spin liquid (KSL) phase with ITO to the partially polarized phase.

Experimentally, there is great interest in the possibility of ITO revival in Kitaev materials near field-suppressed magnetic order [30,31]. Such questions have also inspired phenomenological studies of topological phases and excitations in relevant matter Chern-Simons field theories [32,33]. However, before attempting a microscopic treatment of field-tuning ITO in Kitaev materials, it is important to first understand the behavior in the Kitaev limit.

We present here a microscopic derivation and analysis of an effective topological gauge theory for the isotropic Kitaev model subjected to (001) Zeeman and (Haldane mass) three-spin perturbations. We use a recent Chern-Simons spin fermionization scheme [34] valid for arbitrary 2D lattices that avoids enlargement of the local Hilbert space, with the fermions attached in specific ways to two Abelian Z_2 gauge fields, respectively, on the lattice and dual lattice. Our analysis gives a complete characterization of low-field ITO as well as intermediate field phases in the AFM Kitaev model starting from a microscopic theory.

Our main results are as follows (see Table I): At low fields, in both FM and AFM cases, the fusion rules, threefold GSD on the torus, $\gamma = \ln 2$, and $c = 1/2$ substantiate the phase has ITO, proving Kitaev's original proposal. For the FM case, ITO is lost beyond a critical field h_c and the transition is to a topologically trivial partially polarized state characterized by $\gamma = c = 0$. For the AFM case, the system transitions at h_{c1} (appreciably larger than h_c of the corresponding FM case) to a chiral Abelian phase with fermion quasiparticles. This phase which persists at intermediate fields $h_{c1} < h < h_{c2}$ has trivial

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TABLE I. The table gives a complete characterization of the different field-tuned topological phases of both the FM and AFM Kitaev model.

	Isotropic AFM Kitaev model →			Isotropic FM Kitaev model →	
Degeneracy	Three fold GSD	No GSD		Three fold GSD	
Entanglement Entropy (γ)	ln 2	0		ln 2	
Chiral central charge in the edge theory (c)	1/2	1/2		1/2	
Phase	ITO $\{1, \psi, \sigma\}$	An abelian phase with trivial topological order and fermionic excitations	Partially polarized phase	ITO $\{1, \psi, \sigma\}$	Partially polarized phase
field (h) →	0	h_{c1}	h_{c2}	0	h_c → field (h)

topological order, i.e., $\gamma = 0$, no GSD, and $c = 1/2$, and belongs to the class of a $p + ip$ superconductor where fermion number parity is not gauged. Significantly, *none* of these facts about the intermediate phase can be captured by parton mean field theory, and neglecting gauge fluctuations leads to erroneous results such as sign reversal of half-quantized thermal Hall response [23] at h_{c1}. Beyond h_{c2}, the partially polarized phase appears like the FM case.

We begin with the following SO(3) Majorana fermionization of the spin-1/2 ($\mathbf{S} = \boldsymbol{\sigma}/2$) [35]:

$$\sigma_i^x = -i\eta_i^y \eta_i^z, \quad \sigma_i^y = -i\eta_i^z \eta_i^x, \quad \sigma_i^z = -i\eta_i^x \eta_i^y. \quad (1)$$

The singlet operators $\gamma_i = -i\eta_i^x \eta_i^y \eta_i^z$ anticommute with all fermions except with the η_i fermions at the same site i and act as Jordan-Wigner (JW) strings [36,37]. Field theoretically, the 2D JW transformation [38–41] is implemented by Chern-Simons (CS) flux attachment [42–45], but on the lattice, consistency is achieved only if local one-to-one face-vertex correspondence is satisfied [46]. However, enlarging the gauge fields by including the dual lattice allows construction of consistent mutual CS theories on arbitrary 2D lattices [47–50] with Lagrangian (see Supplemental Material (SM) [51] for details)

$$\mathcal{L}_{CS} = \frac{k}{4\pi}[\xi_{f^*e^*}^* a_{e^*}^* A_v \delta_{f^*v} + D_{v^*e^*}^* a_{v^*}^* A_e \delta_{ee^*} - \partial_0 a_{e^*}^* A_e \delta_{ee^*}]$$
$$- \frac{k}{4\pi}[\xi_{fe} A_e a_{v^*}^* \delta_{fv^*} + D_{ve} A_v a_{e^*}^* \delta_{ee^*} - \partial_0 A_e a_{e^*}^* \delta_{ee^*}], \quad (2)$$

where ξ_{fe} and D_{ve} are, respectively, the lattice analogs of curl and gradient [46] operations and "$*$" everywhere labels the dual lattice. The δ functions are defined as follows: $\delta_{e,e^*} = 1$ if e and e^* are links dual to each other, and zero otherwise, and similarly for δ_{fv^*}, etc. Hereafter, we make the gauge choice that all the temporal components of the electromagnetic fields are zero, i.e., $A_v = a_{v^*}^* = 0$. We fermionize the spin operators, $\sigma_i^\alpha = \gamma_i \eta_i^\alpha$, using the JW string, $\gamma_i = e^{i\int_C^i (B_e + A_e)}$, where $B_e = \frac{k}{2}(a_{e1^*}^* + a_{e2^*}^* + a_{e3^*}^* + a_{e4^*}^*)$ is a specific combination of dual lattice gauge fields [34] enclosing the link e, chosen to

satisfy the spin commutation relations.[1] $\mathcal{C}$ denotes a path that starts from some reference point and ends at lattice site i. We choose both A_e and $a_{e^*}^*$ as Z_2,— gauge invariance constraints models we study to those conserving fermion and vortex number parity. From the property $\sigma_i^2 = 1$, we additionally require $e^{i\int_e 2B_e^i} = 1$. For Z_2 gauge fields, two-body spin interactions on any link $e = (ij)$ have a simple form:

$$\sigma_i^\alpha \sigma_j^\beta = \eta_i^\alpha e^{i(A_e + B_e)} \eta_j^\beta. \quad (3)$$

Comparing with the Hermitian conjugate of Eq. (3) and using the anticommutation property of the η fermions, we obtain the constraint $e^{iB_e} = e^{i\frac{(2n+1)\pi}{2}}$ with n an integer. The conditions $e^{2iB_e} = -1$ and $e^{i\int_e 2B_e} = 1$ ensure the correct spin statistics [51]. As a corollary of Eq. (3), for every link $e = \langle ij \rangle$, $\langle \gamma_i \gamma_j \rangle = e^{i(A_e + B_e)}$. We choose the level $k = 1$ which describes a trivial state [34].

We now proceed to implement Chern-Simons fermionization of the isotropic Kitaev model:

$$\mathcal{H}_K = -K \sum_{\langle pq \rangle \in \vartheta - \text{links}} \sigma_p^\vartheta \sigma_q^\vartheta. \quad (4)$$

Here p, q are the vertices associated with the corresponding link ϑ, with $\vartheta = x$, y, or z. Using the SO(3) representation of Eq. (1), we write $\sigma_i^x \sigma_j^x = i(-i\eta_i^y \eta_j^y)\eta_i^z \eta_j^z = ie^{iA_e^x}\eta_i^z \eta_j^z$ and $\sigma_i^y \sigma_j^y = i(-i\eta_i^x \eta_j^x)\eta_i^z \eta_j^z = ie^{iA_e^y}\eta_i^z \eta_j^z$, where $e^{iA_e^x} = -i\eta_i^y \eta_j^y$ for the x link and $e^{iA_e^y} = -i\eta_i^x \eta_j^x$ for the y link are chosen such that commutation relations of these η bilinears are preserved. We perform a Hubbard-Stratonovich decoupling of the four-fermion interaction on the z bond following Ref. [23], $-\sigma_i^z \sigma_j^z = -(\gamma_i \eta_i^z)(\gamma_j \eta_j^z) = -m_i \gamma_j \eta_j^z - m_j \gamma_i \eta_i^z + m_i m_j - i\delta\eta_i^z \eta_j^z - \gamma_i \gamma_j \Delta + \delta\Delta$, with the mean fields $m_i = \langle \gamma_i \eta_i^z \rangle$, $\delta = \langle i\gamma_i \gamma_j \rangle$ and $\Delta = \langle i\eta_i^z \eta_j^z \rangle$. We now introduce the following time-reversal symmetry-breaking terms that preserve fermion number parity: $\mathcal{H}_Z = -h \sum_i \sigma_i^z$, is a Zeeman (001) perturbation and $\mathcal{H}_\kappa = -\kappa \sum_{j,k,l} \sigma_j^x \sigma_k^y \sigma_l^z$, corresponds to the scalar spin chirality (Haldane mass term). The (001) orientation has garnered interest recently because of enhanced field-robustness of the KSL [22,23]. For small deviations of the field direction from (001), the Haldane mass term—necessary for chiral phases—is also generated. The model now has two parameters h and κ; however, $\mathcal{H}_\kappa$ commutes with the vison (flux) operators and therefore will not cause the vison fluctuations necessary to bring about the topological transitions; also see SM [51]. The mean field Hamiltonian in momentum space is [23] (see SM [51] for details)

$$\mathcal{H}_K = \sum_{\bar{k}} \Psi_{\bar{k}}^\dagger \mathcal{M}_K \Psi_{\bar{k}} + NK_z \delta\Delta + NK_z m^2, \quad (5)$$

with $\Psi_{\bar{k}}^T = [i\gamma_{\bar{k}A} \quad i\gamma_{\bar{k}B} \quad \eta_{\bar{k}A}^z \quad \eta_{\bar{k}B}^z]$ and

$$\mathcal{M}_K = \begin{bmatrix} 0 & \frac{iK\Delta}{2} & -\frac{i(Km+h)}{2} & 0 \\ -\frac{iK\Delta}{2} & 0 & 0 & -\frac{i(Km+h)}{2} \\ \frac{i(Km+h)}{2} & 0 & \frac{W}{2} & \frac{-iKf}{2} \\ 0 & \frac{i(Km+h)}{2} & \frac{iKf^*}{2} & -\frac{W}{2} \end{bmatrix}, \quad (6)$$

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[1] The normalization in B_e is $\frac{k}{2}$, which differs from Ref. [34] by a factor of 2 and corrects it.
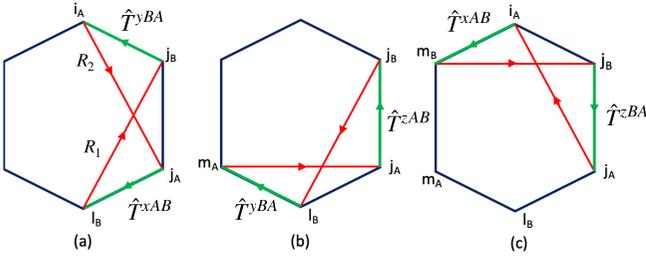
FIG. 1. The configurations that give CS terms. The green lines represent the $\hat{T}$ matrices and red lines are associated with Green's function elements.

where $W^2 = \kappa^2 \{\delta \sin(0.5k_x + 0.5\sqrt{3}k_y) + \delta \sin(0.5k_x - 0.5\sqrt{3}k_y) - \sin(k_x)\}^2$ and $f = e^{-ik_1} + e^{-ik_2} + \delta$, and $m_{i_A} = m_{i_B} = m$. N is the number of unit cells, A and B are sublattice indices, and $k_1 = \vec{k}.\hat{n}_1$, $k_2 = \vec{k}.\hat{n}_2$, where $\hat{n}_1$ and $\hat{n}_2$ are the unit vector along the x and y links, respectively. The mean fields are calculated self-consistently using Eqs. (5) and (6) [23]. The low-lying bands are chiral, with the Dirac points located at $\vec{k}_0 = (k_0, 0)$, where

$$\cos \frac{k_0}{2} = -\frac{1}{2\Delta}\{\Delta\delta + (Km + h)^2\} = -\frac{\xi}{2}. \quad (7)$$

An advantage of this mean field scheme over earlier parton mean field studies, e.g., Refs. [27,28], is the avoidance of Hilbert space enlargement that can introduce extra topological phases. Most of the numerical studies [17,20,25] broadly show only two field-tuned transitions like we find here. See the SM [51] for a validation of our mean field choice by benchmarking with extreme limit results.

To obtain the effective gauge action, we first integrate out the massive fermions, yielding

$$S_e = -\frac{1}{2}\text{Tr}\ln\{G_0^{-1} + \hat{T}\} + S_{CS} + NK_z(\delta\Delta + m^2). \quad (8)$$

Here $G_0^{-1} = (-i\omega_n + \mathcal{M}_K)$ is the inverse Green's function corresponding to the mean-field Hamiltonian (i.e., without the gauge fluctuations). The matrix $\hat{T}$ contains the gauge fluctuation parts of both Kitaev and the scalar spin chirality terms [51].

Expanding the logarithm term in Eq. (8), we first get non-trivial contributions coming from the second order for the links terms. For the $x(y)$ bonds, we get Maxwell-type electric field terms, $\mathcal{L}_{x(y)} = E_c^{-1}(\dot{A}_e^{x(y)})^2$, with $E_c \approx \frac{W}{2}\frac{[\Delta^2 + (m + \frac{h}{K})^2]^2}{(m + \frac{h}{K})^2}$, and for the z bonds, $\mathcal{L}_z = \delta^2 E_c^{-1}(\dot{A}_e^z + \dot{B}_e^z)^2$. As a high field benchmarking, we check that $E_c \sim h^3/K^2$ grows rapidly with h, while the order parameter δ vanishes, resulting in an effectively one-dimensional character of the low excitations in the x-y backbone reported earlier [22,34]. Chern-Simons terms arise from the processes shown in Fig. 1. For example, the configuration in Fig. 1(a) gives [51]

$$\Delta S_{CS} = \frac{i}{4\pi}\text{sign}(\xi)\,\text{sign}(W') \sum_{\langle\langle ee'\rangle\rangle} \int d\tau A_e^x \frac{\partial A_{e'}^y}{\partial \tau}, \quad (9)$$

with $e = j_A \to l_B$ and $e' = i_A \to j_B$. It is easy to obtain similar CS contributions from the diagrams in Figs. 1(b) and 1(c).

Note that the two link fields in our CS action are separated by precisely one link and do not share a vertex. In our formalism, the z links are associated with the gauge fields $A_e + B_e$. Consequently, diagrams containing one z link [e.g., Figs. 1(b) and 1(c)] also generate a correction to the mutual CS interaction:

$$\Delta S_{MCS} = \frac{i}{4\pi}\text{sign}(\xi)\,\text{sign}(W') \sum_{\langle\langle ee'\rangle\rangle} \int d\tau A_e^y \frac{\partial B_{e'}^z}{\partial \tau}. \quad (10)$$

In our model, the sign of effective gap W' does not change unless κ changes sign. However, $\text{sign}(\xi)$ is controlled by the strength of the Zeeman field [51]. In the AFM case, ξ changes sign at h_{c1}. Surprisingly, the sign change of the CS level (chirality) comes not from band inversion but reversal of a velocity component [23].

In the remaining part of the paper, we study the nature of the topological phases in different field regimes. For this purpose, we go over to the continuum limit. The lattice gauge field A_e is a line integral of the vector potential $\int_e \vec{A}.d\vec{l}$. The lattice sum passes over to the continuum limit as $\sum_{links} = \frac{1}{V}\int d^2x$, where $V = 3\sqrt{3}a^2/2$ is the unit cell volume. The continuum limit CS theory has two Abelian Z_2 gauge fields and effective (field-dependent) level matrix:

$$K = \begin{bmatrix} \text{sign}(\xi)\,\text{sign}(W') & 1 + \text{sign}(\xi)\,\text{sign}(W') \\ 1 + \text{sign}(\xi)\,\text{sign}(W') & 0 \end{bmatrix}. \quad (11)$$

At low fields ($0 \leqslant h < h_{c1}$), $\text{sign}(\xi)\,\text{sign}(W') = 1$, so $K_{low} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$, which is different from a Z_2 spin liquid for which $K_{Z_2} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$. In the AFM case for $h_{c1} \leqslant h < h_{c2}$, we have $\text{sign}(\xi)\,\text{sign}(W') = -1$ and $K_{inter} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$. At even higher fields, $h > h_{c2}$ corresponding to the partially polarized phase, $\text{sign}(\xi)\,\text{sign}(W') = 0$, and we revert to the bare trivial K matrix. For the FM case, a single phase transition occurs at some h_c, from K_{low} to the trivial one. Having obtained the level matrices in the different field regimes, we now describe the nature of these phases.

Consider first Z_2 topological order with level K_{Z_2}. The (Abelian) quasiparticles (e, m, ϵ) have the general form $X = e^{i\int_C l_X^\alpha A_{\alpha\mu} dx^\mu}$, where $l_X^\alpha \in Z$, $A_{1\mu} = A_\mu$, and $A_{2\mu} = a_\mu^*$. The quasiparticle charges are $l_e = (1\ 0)^T$, $l_m = (0,\ 1)^T$, and $l_\epsilon = (1,\ 1)^T$. The fusion rules and braiding statistics follow from the canonical commutation relations $[A_\mu(x), a_\nu^*(x')] = 2\pi i\epsilon_{\mu\nu}K_{Z_2}^{-1}\delta^2(x - x') = \pi i\epsilon_{\mu\nu}\delta^2(x - x')$. The Wilson loop $e^{i\oint_C \vec{A}.d\vec{l}}$ gives $+1$ when it encloses the trivial quasiparticle **1** or the e quasiparticle, and -1 otherwise. Similarly, the Wilson loop $e^{i\oint_C \vec{a}^*.d\vec{l}}$ is $+1$ when it encloses **1** or the m quasiparticle, and -1 otherwise. The ϵ particle corresponds to both the eigenvalues being -1. Any Abelian phase and its topological properties can be completely described by the Abelian K matrix. But that is not true, in general, for any non-Abelian phase like ITO, which we will describe now. For $K = K_{low}$, Z_2 order is clearly lost since the above fusion rules are no longer satisfied.

For an anyon model which is symmetric under some permutations of their topological charges (e.g., $e \leftrightarrow m$ for Z_2), one can describe a non-Abelian phase by introducing twist defects [52–55] such as dislocations. This is related to a
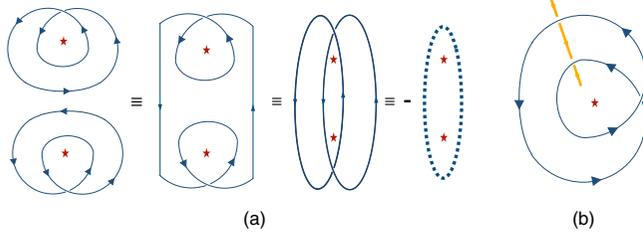
FIG. 2. Illustration of fusion rules for twist defects. The blue and yellow lines represent the string $e^{i \int \vec{a}^* \cdot d\vec{l}}$ and $e^{i2 \int \vec{a}^* \cdot d\vec{l}}$, respectively. In (a), fusion of two closed strings that wind twice are shown. The dotted line measures the fermion number parity within. The fusion between the quasiparticles ψ and σ is shown in (b).

local symmetry operation $G = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ on K_{Z_2} that amounts to interchanging the two species of gauge fields locally. For Z_2 topological order, it is known that such twist defects (σ) are non-Abelian ITO anyons [52,56].

Now we show that the structure of K_{low} naturally admits a twist defect that satisfies the ITO fusion rules—specifically, the diagonal entry in K_{low} introduces a single twist that breaks e and m exchange symmetry. We show that the Wilson loop $\Gamma = e^{i \oint_{c_2} \vec{a}^* \cdot d\vec{l}}$, where the closed loop c_2 winds twice (say anticlockwise) with only one crossing plays the role of detector of a twist defect σ. This is analogous to the double winding string detectors of twist defects $\sigma_\pm$ in Z_2 spin liquids [52]—the main difference in the Z_2 case is the time-reversal symmetry that allows two distinct twist defects of opposite chirality, and respective eigenvalues $+i$ or $-i$ for the corresponding double winding detector. In our case, only one of these eigenvalues is realized because all defects have the same chirality. Mathematically, the imaginary eigenvalue for the double winding loop comes from the nontrivial commutation relation $[a^*_\mu(x), a^*_\nu(x')] = 2\pi i \epsilon_{\mu\nu} K^{-1}_{low} \delta^2(x - x') = -\frac{\pi}{2} i \epsilon_{\mu\nu} \delta^2(x - x')$ in our model. On the other hand, because of the presence of the diagonal term, e and m charges cannot be defined consistently. We now obtain the fusion rules of these twist defects. In the absence of other excitations, for two closed strings that wind twice, say Γ_1 and Γ_2, we have $\Gamma_1 \Gamma_2 = -1$ in Fig. 2(a), where the minus sign comes from the presence of two nontrivial crossings, and the remaining dotted curve counts the fermion number parity within. The ψ fermionic quasiparticle can be realized with the string, $e^{i2 \int \vec{a}^* \cdot d\vec{l}}$, and introducing a ψ into any double winding loop of the type Γ simply changes the fermion number parity. The fermion number parity can be ± 1, i.e., modulo 2, the charge within is 1 or ψ. In Fig. 2(b), introduction of a ψ (yellow line) into Γ clearly does not change the sign, consistent with the fusion rule $\sigma \times \psi = \sigma$.

Now we define the Majorana fermion by an open string $C_j = e^{i \int \vec{a}^* \cdot d\vec{l}}$ that loops around a twist defect at j and all Majorana strings thus defined are to have the same end points. Two such strings (say, C_j and C_{j+1}) with the same end points, when braided, result in the change in the orientation of their intersection, yielding the braiding relations $C_j \to C_{j+1}$, $C_{j+1} \to -C_j$, consistent with Refs. [52,57]. The total charge upon fusion is $-iC_jC_{j+1}$, taking values either $+1$

for **1** or -1 for ψ, respectively. The anyon sector $\{1, \psi, \sigma\}$ gives ITO.

In anyon-CFT correspondence [58–61], it is well-known that these twist fields in $1 + 1$ dimensions are either holomorphic or antiholomorphic, which is the case for our model. From the edge theory, one can also calculate the conformal weight of these fields and the braiding statistics of ITO. The GSD on the torus for this ITO phase is three due to three superselection sectors discussed above. The topological entanglement entropy (γ) can be expressed as $\gamma = \ln D$, where D is the total quantum dimension. In case of the ITO, $D^2 = \sum_\alpha d_\alpha^2$ with $d = 1$ for vacuum and ψ and $d = \sqrt{2}$ for quasiparticle σ, hence $\gamma = \ln 2$ in ITO as expected. On the other hand, the field regime $h_{c1} \leqslant h < h_{c2}$ corresponds to a chiral Abelian phase with trivial topological order and fermionic bulk excitations, so $D = 1$, $\gamma = 0$. Note that in the ITO phase, strings composed of the gauge field A_μ do not have any nontrivial commutation relation with the σ or ψ quasiparticles of ITO. On the other hand, in the intermediate phase of the AFM Kitaev model, they are the sole gauge fields and give the parity and time-reversal symmetry-breaking CS term in the theory. The gauge sector gives a nontrivial contribution to the central charge in this regime, and is associated with the nonzero vison Chern number. It is understood [57,60] that the $p + ip$ superconductor supports intrinsic non-Abelian Ising anyons upon gauging the fermion number parity. In our theory, the fermion number parity is associated with the a^* dynamical gauge field and A is associated with the vortex number parity. The intermediate field phase is deconfined through the A dynamical gauge field which is associated with vortex number parity.

Chiral central charge. From the bulk-edge correspondence [62–64], the effective edge theories of K_{low} imply the following Kac-Moody algebra of chiral bosons (ϕ), $[\phi_I(x), \partial_y \phi_J(y)] = \pm 2\pi i K^{-1}_{low I,J} \delta(x - y)$. The fact that the number of positive and negative eigenvalues are equal results in same number of left movers and right -movers on the edge, and corresponds to zero net chirality ($c = 0$). Thus, the central charge in the ITO phase, coming solely from bare chiral Majoranas is $\frac{1}{2}$ [64]. On the other hand, the intermediate phase of the AFM Kitaev model has an effective Abelian CS theory of level $k = -1$ and leads to $c = -1$ from the edge theory [64], while for the bare chiral Majoranas, the Chern number (-1) is flipped with respect to the low field phase [23]. The resultant chiral central charge in this phase taking into account both the Majorana and gauge sectors is $c = c_{Majorana} - c_{gauge} = \frac{1}{2}$.

Discussion. In summary, we presented a microscopic derivation of Abelian lattice Chern-Simons gauge theories for the FM and AFM Kitaev models subjected to time-reversal symmetry-breaking Zeeman (001) and scalar spin chirality perturbations. We obtained a comprehensive understanding of the topological phases and emergent excitations in different field regimes. In the low field phase, we proved the long proposed ITO by constructing the intrinsic non-Abelian σ anyons as twist defects with a specific chirality. In the AFM case, the intermediate field phase associated with trivial topological order was found to be chiral with $c = 1/2$, (same as the ITO phase, with no sign change) and the same half-quantized thermal Hall response as the ITO phase (see SM [51] and Ref. [65] for details). Experimentally, the transition to the intermediate phase could be detected as a sharp drop in the

plaquette flux value from unity [51] but lacking the 1D excitations of the polarized high-field phase. However, extrinsic Majorana anyons could still be induced in such a phase by freezing spins locally [66,67]. Our technique provides a way for understanding other open questions, e.g., the possibility of field-revived ITO in the simultaneous presence of competing ITO degrading perturbations such as Zeeman and exchange interactions.

Our CS fermionization approach is generally applicable to 2D spin-1/2 systems with fermion number parity symmetry and could be useful in understanding the effects of perturbations on the stability and phase transitions in different spin liquid systems. From a practical perspective, our approach could be useful for realizing fault-tolerant non-Abelian qubits in an Abelian gauge theory setting.

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